Improved bounds for the zeros of the chromatic polynomial

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Combinatorial Algorithmic and Probabilistic Aspects of Partition Functions
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Joint work with
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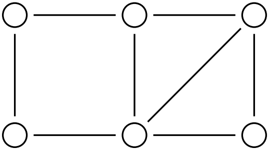
Graph colouring



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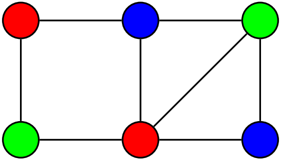

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$f : V(G) \to \{1, \ldots, k\}$ s.t. $f(u) \neq f(v)$ whenever $uv \in E(G)$

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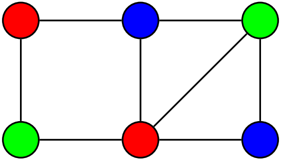

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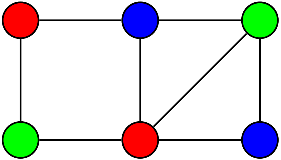

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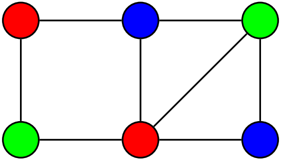

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- Originally introduced to approach four colour problem
- Examples
  - $C_{k_r}(k) = k(k-1)(k-2)\cdots(k-r+1)$
  - $C_{n\text{-vertex tree}}(k) = k(k-1)^{n-1}$

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Want FPTAS / FPRAS for

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- exactly $\forall z \in \mathbb{C} \setminus \{0, 1, 2\}$. Jaeger-Vertigan-Welsh 1990
- approximately $\forall z$ s.t. $|z - 1| > 1$ Bencs-Huijben-Regts '22

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[GSV15]

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Conjecture $\exists$ FPTAS for $C_G(k)$ provided $k > \Delta(G)$

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For $k \in \mathbb{N}$

- FPRAS for $k \geq 2\Delta$ Jerrum 1994
- FPRAS for $k > (\frac{11}{6} - \varepsilon)\Delta$ Vigoda 2006, CDMPP 2019
- FPTAS for $k \geq 2\Delta$ Liu-Sinclair-Srivastava 2019

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Improved bounds for small Δ (Fialho-Juliano-Procacci 2024+)

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Conjecture: $C_G(z) \neq 0$ if $\Re(z) > \Delta(G)$ Sokal 2003

Implies

Conjecture: $\exists$ FPTAS for $C_G(k)$ provided $k > \Delta(G)$
[1990's, Frieze-Vigoda]

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$$C_G(z) = a_0 z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^n a_n$$

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Theorem (Whitney 1932)

a_i = number of broken-circuit free sets of size i in G

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Broken circuit free sets (BCF sets)

Fix an ordering of E. We say $A \subseteq E$ is broken-circuit free if

- A is a forest, and
- each $e \in E \setminus A$ is not the largest edge in the unique cycle of $A + e$ (when it exists)

Note: number of BCF sets is independent of edge ordering!

Example

We work with a simple transformation:

$$C_G(z) = a_0 z^n - a_1 z^{n-1} + \cdots + (-1)^n a_n$$
$$B_G(z) = a_0 + a_1 z + \cdots + a_n z^n = z^n C_G(-z^{-1}) = \sum_{F \subseteq E \text{ BCF}} z^{|F|}$$

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Enough to show that whenever $\Delta(G) \leq \Delta$ and $|z| \leq 1/K\Delta$

$$\left| \frac{B_G(z)}{B_{G-u}(z)} \right| \in [1-a, 1+a]$$

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for $K = 6.91$ and $a = 0.32$

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Enough to complete the induction ...

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Equivalently $\sum_{\substack{F+uS \text{ is} \\ \text{BCF tree}}} z^{|F|} B_{G-u-S-V(F)} \approx B_{G-u}$

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For final inequality we introduce and bound

$$T_{G,v_1,v_2}(x) = \sum_{\substack{T \text{ tree:} \\ v_1,v_2\in V(T)}} x^{|T|} \quad \text{and bound} \quad T_{G,v_1,v_2}\left(\frac{\ln\alpha}{\alpha\Delta}\right) \le \frac{\alpha\ln\alpha}{\Delta}$$

Conclusion

$C_G(z) \neq 0$ whenever $\Delta(G) \leq \Delta$ and $|z| \geq 5.93\Delta$

- Try to improve on 5.93
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If girth$(G) \geq g$ and $\Delta(G) \leq \Delta$, can replace 5.93 with $K_{g,\Delta}$

$K_{3,3} = 4.55$ $K_{3,4} = 4.90$ $K_{3,5} = 5.11$
$K_{4,3} = 3.84$ $K_{5,3} = 3.48$ $K_{100,3} = 2,49$

Fialho-Juliano-Procacci 2024+

Further Results

Forest generating polynomial of $G = (V, E)$

$$F_G(z) = \sum_{F \subseteq E \text{ forest}} z^{|F|}$$

- Also called partition function of arboreal gas model
- Our methods extend here, but can go further using a different recursion

$F_G(z) \neq 0$ whenever $\Delta(G) \leq \Delta$ and $|z| \leq 1/(2\Delta)$

Jenssen-Patel-Regts 2024

- Cannot replace $1/(2\Delta)$ with $1/\Delta$ due to Δ-multi edge.
- Can we get close to $1/\Delta$?