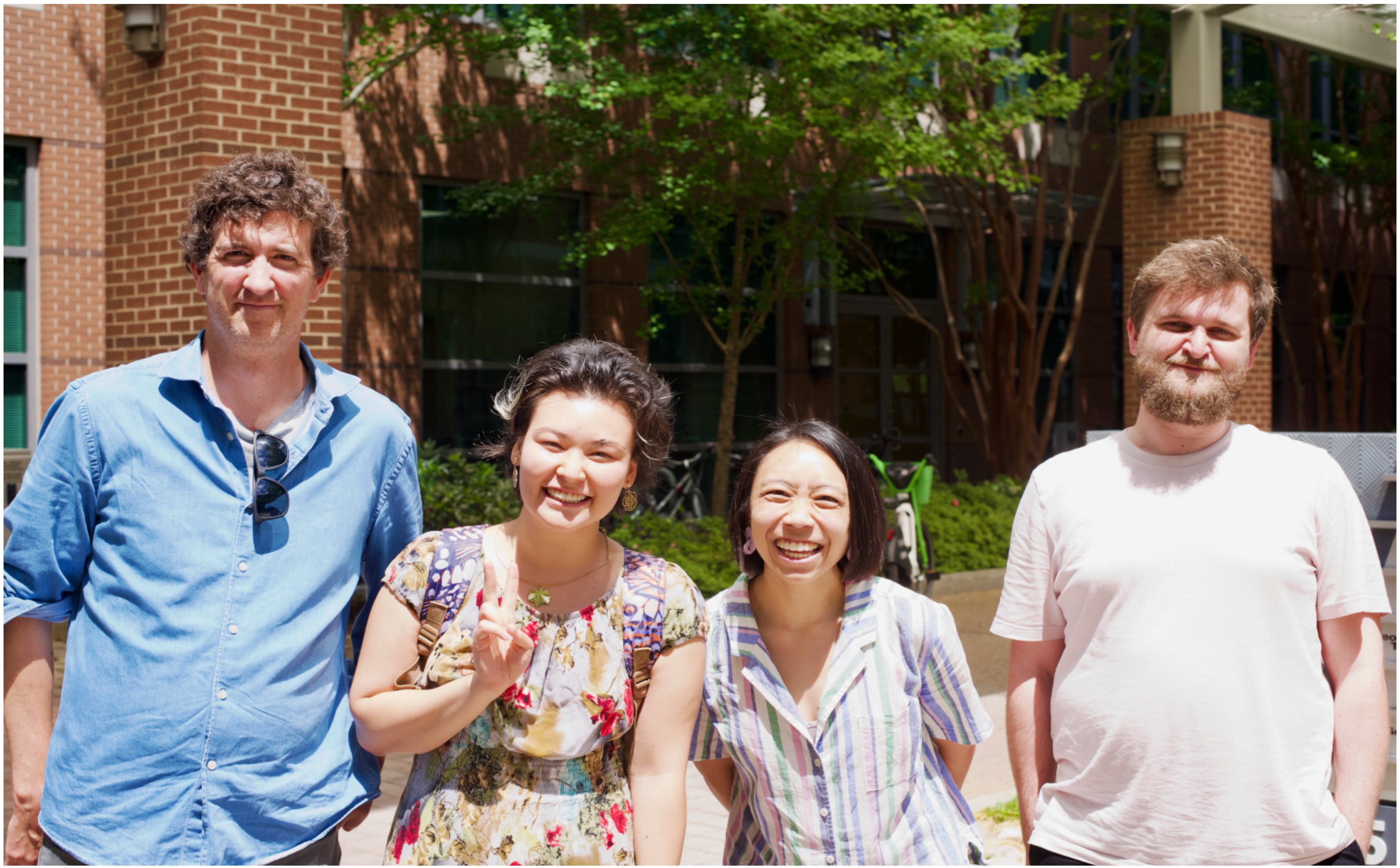
Fast and Slow Mixing of the Kawasaki Dynamics on Bounded-Degree Graphs

Aiya Kuchukova, **Marcus Pappik**, Will Perkins and Corrine Yap

Workshop on the Combinatorial, Algorithmic and Probabilistic Aspects of Partition Functions

Meet the Authors



Ising Model

Ferromagnetic Ising model: a model of spontaneous magnetization

- given graph G, edge interaction $\beta \in \mathbb{R}_{\geq 0}$, external field $\lambda \in \mathbb{R}_{\geq 0}$
- state space $\Omega = \{\sigma : V(G) \to \{+1, -1\}\}$
- distribution $\mu_{G,\beta,\lambda}(\sigma) \propto \lambda^{|\sigma|^+} \mathrm{e}^{\beta m(\sigma)}$ on Ω
- partition function $Z_G(\beta, \lambda) = \sum_{\sigma \in \Omega} \lambda^{|\sigma|^+} \mathrm{e}^{\beta m(\sigma)}$

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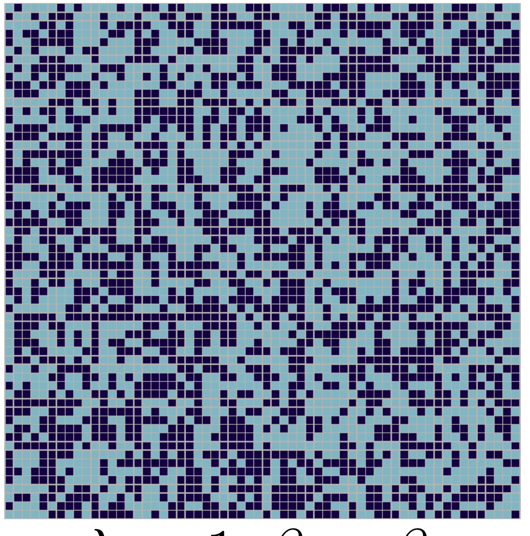
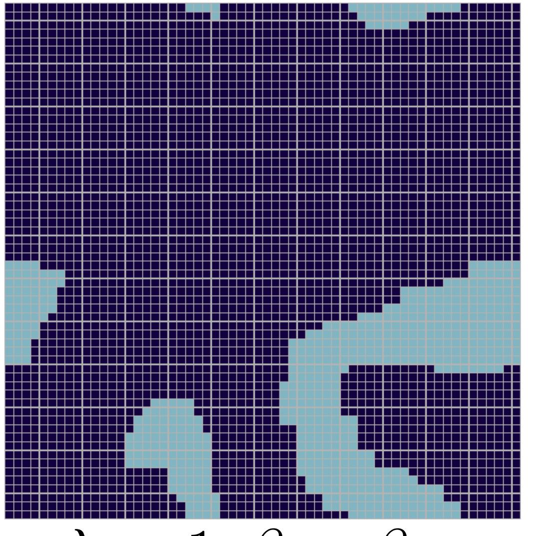
Phase Transition: **small quantitative** change of parameters leads to **large qualitative** change of the entire system.



$\lambda = 1, \beta < \beta_u$



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Ising Model: Phase Transitions on $\mathcal{G}_\Delta$

Probabilistic: uniqueness vs. non-uniqueness on $\mathbb{T}_\Delta$

- uniqueness on $\mathbb{T}_\Delta$ iff $\beta < \beta_u(\Delta)$ or $\lambda \notin \left[\frac{1}{\lambda_u(\Delta,\beta)}, \lambda_u(\Delta, \beta)\right]$

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Analytic: absence of roots of $\lambda \mapsto Z_G(\beta, \lambda)$ in complex domains

- all roots of $\lambda \mapsto Z(\beta, \lambda)$ are on the unit circle [Lee and Yang ('52)]
- no root near $\mathbb{R}_{\geq 0}$ if $\beta < \beta_u(\Delta)$ [Peters and Regts ('20)]
- no tight regime of zero-freeness under pinnings (absolute zero-freeness)

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Computational: NP-hardness of approximation vs. efficient algorithms

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Dynamical: slow vs. fast convergence of a 'natural' Markov chain to $\mu_{G,\beta,\lambda}$

Mixing of Glauber dynamics:

- rapid mixing for $\beta < \beta_u(\Delta)$ [Mossel and Sly ('13)]
- slow mixing for $\beta > \beta_u(\Delta), \lambda = 1$ [Dembo and Montanari ('10)]
- absolute zero-freeness implies rapid mixing [Chen, Liu and Vigoda ('21)]

Fixed-Magnetization Ising Model

magnetization: for graph G and $\sigma \in \Omega$ define $\eta(\sigma) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \sigma(v)$

fixed-magnetization (or canonical) Ising model:

- given graph G, edge interaction $\beta \in \mathbb{R}_{\geq 0}$, magnetization $\eta \in [-1, 1]$
- state space $\Omega_\eta = \{\sigma : V(G) \to \{+1, -1\} \mid \eta(\sigma) = \eta\}$
- distribution $\hat{\mu}_{G,\beta,\eta}(\sigma) \propto \mathrm{e}^{\beta m(\sigma)}$ on Ω_η
- partition function $\hat{Z}_G(\beta, \eta) = \sum_{\sigma \in \Omega_\eta} \mathrm{e}^{\beta m(\sigma)}$

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Slightly different point of view:

For all $\lambda > 0$ it holds that

$$\hat{\mu}_{G,\beta,\eta}(\,\cdot\,) = \mu_{G,\beta,\lambda}(\,\cdot \mid \eta(\sigma) = \eta).$$

Fixed-Magnetization Ising: Computational Threshold

Computational Threshold (Carlson-Davies-Kolla-Perkins '22)

There is some $\eta_c(\Delta, \beta) > 0$ such that:

- If $\beta < \beta_u(\Delta)$ or $|\eta| > \eta_c(\Delta, \beta)$, then there is an FPRAS for $\hat{Z}_G(\beta, \eta)$ for $G \in \mathcal{G}_\Delta$.
- If $\beta > \beta_u(\Delta)$ and $|\eta| < \eta_c(\Delta, \beta)$, then approximating $\hat{Z}_G(\beta, \eta)$ on $\mathcal{G}_\Delta$ is NP-hard.

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What is the threshold?

$\eta_c(\Delta, \beta)$ is the "largest" expected magnetization of the Ising model at $\lambda = 1$ on any $G \in \mathcal{G}_\Delta$:

$$\eta_c(\Delta, \beta) := \sup_{G \in \mathcal{G}_\Delta} \mathbb{E}_{\sigma \sim \mu_{G,\beta,1}}[\eta(\sigma)]$$

Fixed-Magnetization Ising: Dynamical Threshold

Kawasaki Dynamics: a natural Markov chain for fixed magnet. Ising

- pick a $+1$ and -1 vertex uniformly at random
- swap their spins with probability proportional to the ratio of weights before and after swapping (Metropolis update)

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Conjecture (Carlson-Davies-Kolla-Perkins ’22)

The mixing time of Kawasaki dynamics is polynomial in $|V(G)|$ for all $G \in \mathcal{G}_\Delta$ if and only if $\beta < \beta_u(\Delta)$ or $|\eta| > \eta_c(\Delta, \beta)$. That is, **computational and dynamical threshold coincide** on $\mathcal{G}_\Delta$.

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Support for the conjecture:

- for **fixed-size independent sets** the dynamical and computational threshold coincide [Jain, Michelen, Pham and Vuong ('23)]
- for **fixed-size matchings** we have rapid mixing whenever we have an approx. counting algorithm [Jain and Mizgerd ('24)]

Main Result: Fixed-Magnetization Ising Model

Theorem 1

There are $\eta_a(\Delta, \beta) \geq \eta_u(\Delta, \beta) > \eta_c(\Delta, \beta)$ such that:

(1) If $\beta < \beta_u(\Delta)$ or $|\eta| > \eta_a(\Delta, \beta)$, the mixing time of Kawasaki dynamics is **polynomial** in $|V(G)|$ for all $G \in \mathcal{G}_\Delta$.

(2) If $\beta > \beta_u(\Delta)$ and $|\eta| < \eta_u(\Delta, \beta)$, the mixing time is **exponential** in $|V(G_n)|$ for some sequence $G_n \in \mathcal{G}_\Delta$, $|V(G_n)| \to \infty$.

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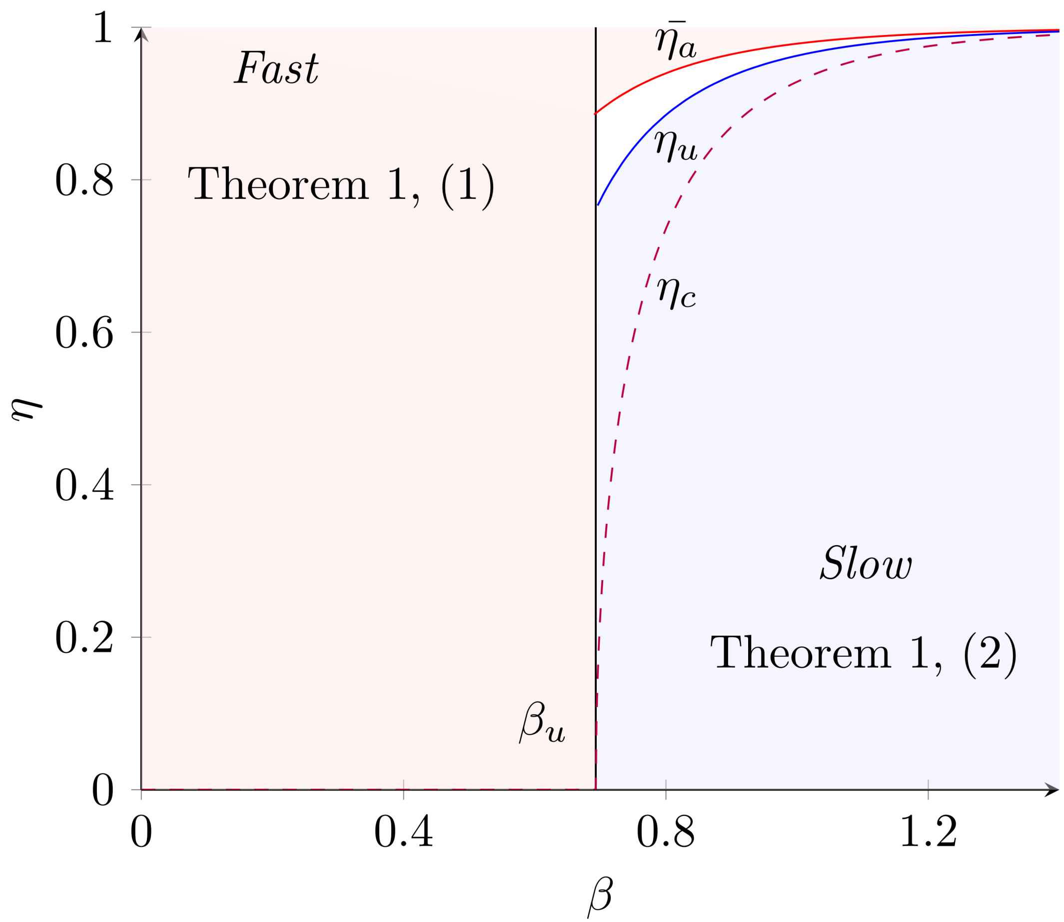

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Absolute zero-freeness:
For all $G \in \mathcal{G}_\Delta$, $S \subset V$ and $\tau : S \to \{-1, +1\}$

$$\lambda \mapsto Z_G^\tau(\beta, \lambda) := \sum_{\substack{\sigma \in \Omega: \\ \sigma_{|S} = \tau}} \lambda^{|\sigma|^+} \mathrm{e}^{\beta m(\sigma)}$$

is zero-free in a neighborhood of every compact $D \subset (\lambda_a(\Delta, \beta), \infty)$.

Part I:
Rapid Mixing

Rapid Mixing from ℓ_∞-Independence

ℓ_∞-independence:
We say a distribution π on $\Omega = \{\sigma : V \to \{-1, +1\}\}$ is C-ℓ_∞-independent if for all $u \in V$ with $\pi(u \mapsto +1) > 0$ it holds that

$$\sum_{v \in V} \left| \pi(v \mapsto +1 \mid u \mapsto +1) - \pi(v \mapsto +1) \right| \leq C.$$

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ℓ_∞-independence $\Rightarrow$ rapid mixing :
If the **fixed magnet. Ising model at magnetization $\eta < 0$** is $O(1)$-ℓ_∞-independent **under every +1-pinning**, then the Kawasaki dynamics mix rapidly for that η.

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Idea: relate ℓ_∞-independence fixed magnet. Ising and general Ising
(adapting a framework by Jain, Michelen, Pham and Vuong ('23))

Establishing ℓ_∞-Independence

Note: For all $\lambda > 0$, pinnings τ and $u, v \in V$ we have

$$\hat{\mu}_\eta^\tau(v \mapsto +1) = \mu_\lambda^\tau(v \mapsto +1) \cdot \frac{\mu_\lambda^\tau(\eta \mid v \mapsto +1)}{\mu_\lambda^\tau(\eta)}$$

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Observation:

To show $O(1)$-ℓ_∞-independence for the fixed magnet. Ising under any pinning τ, it suffices to find some $0 < \lambda$ such that

- μ_λ^τ satisfies $O(1)$-ℓ_∞-independence and
- μ_λ^τ has a **stable magnetization**: for all $u, v \in V$ it holds that

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If $\eta < -\eta_a$, then there is some $\lambda < 1/\lambda_a$ with $\mathbb{E}_{\sigma \sim \mu_\lambda^\tau}[\eta(\sigma)] = \eta$ that satisfies both conditions.

ℓ_∞-Independence and Stability from Zero-Freeness

$\boldsymbol{\ell_\infty}$-independence of Ising:

Absolute zero-freeness implies ℓ_∞-independence [Chen, Liu and Vigoda '21].

$$\sum_{v \in V} \left| \mu_\lambda^\tau(v \mapsto +1 \mid u \mapsto +1) - \mu_\lambda^\tau(v \mapsto +1) \right| = \lambda \frac{\mathsf{d}}{\mathsf{d}\varepsilon} \log \frac{Z^{\tau, u \mapsto +1}(\lambda + \varepsilon)}{Z^\tau(\lambda + \varepsilon)} \Big|_{\varepsilon = 0}$$

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Stablility of magnetization for Ising:

- try to understand the distribution of $X := |\sigma|^+$ near $\mathbb{E}[X]$

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Edgeworth expansion for $\boldsymbol{X}$:

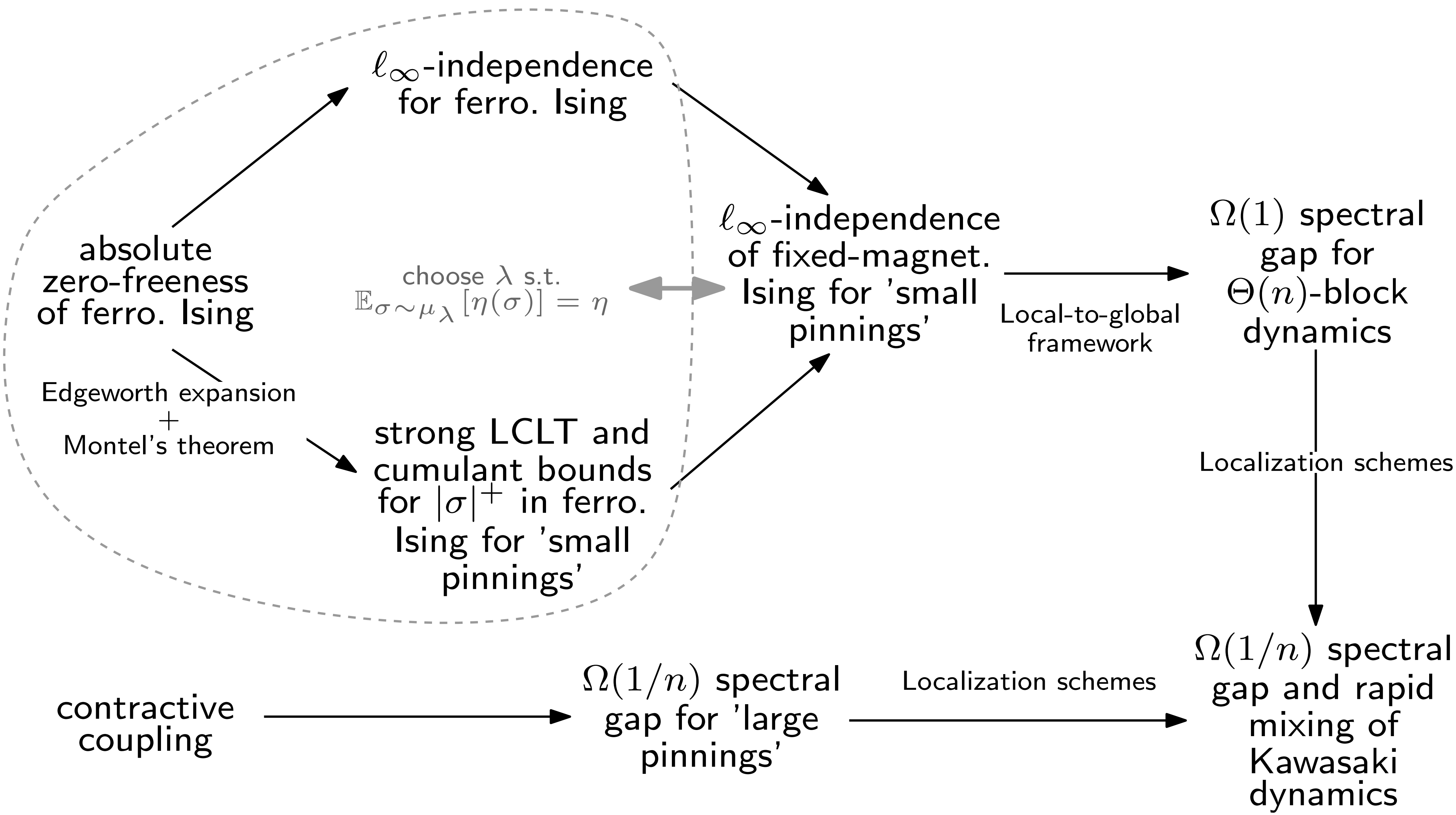
Using the inverse Fourier transformation, we can write the probability mass function of X as

$$f_X(x) = \frac{1}{2\pi} \int \mathrm{e}^{-itx} \exp\left(\sum_{m=0}^{\infty} \kappa_m \frac{(it)^m}{m!} \right) \mathrm{d}t\,,$$

where κ_m is the m^{th} cumulant, given by

$$\kappa_m := \frac{\mathrm{d}^m}{\mathrm{d}t^m} \log \mathbb{E}\left[\mathrm{e}^{itX}\right] \Big|_{t=0} = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \log \frac{Z^\tau(\mathrm{e}^{it}\lambda)}{Z^\tau(\lambda)} \Big|_{t=0}.$$

Proof Overview: Fast Mixing of Kawasaki Dynamics



Part II:
Slow Mixing

Proof Overview: Slow Mixing

non-uniqueness of ferro. Ising on $\mathbb{T}_\Delta$ $\xrightarrow{\text{[GSVY'16]}}$ metastability of ferro. Ising on Δ-regular random graphs

metastability of ferro. Ising on Δ-regular random graphs $\rightarrow$ slow mixing of Glauber dynamics

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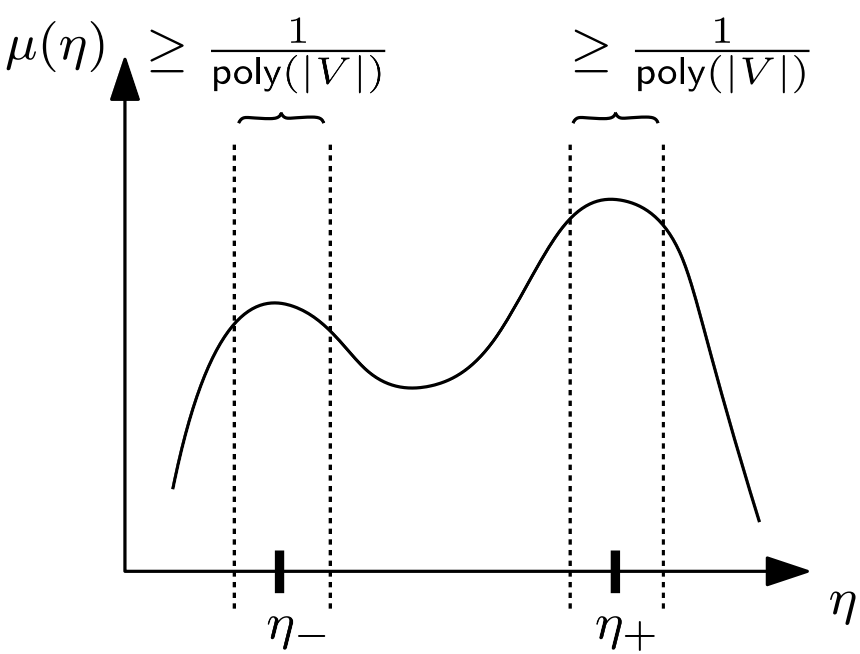
Glauber dynamics:

For $G \sim U(\mathcal{G}_\Delta)$ we have w.h.p.



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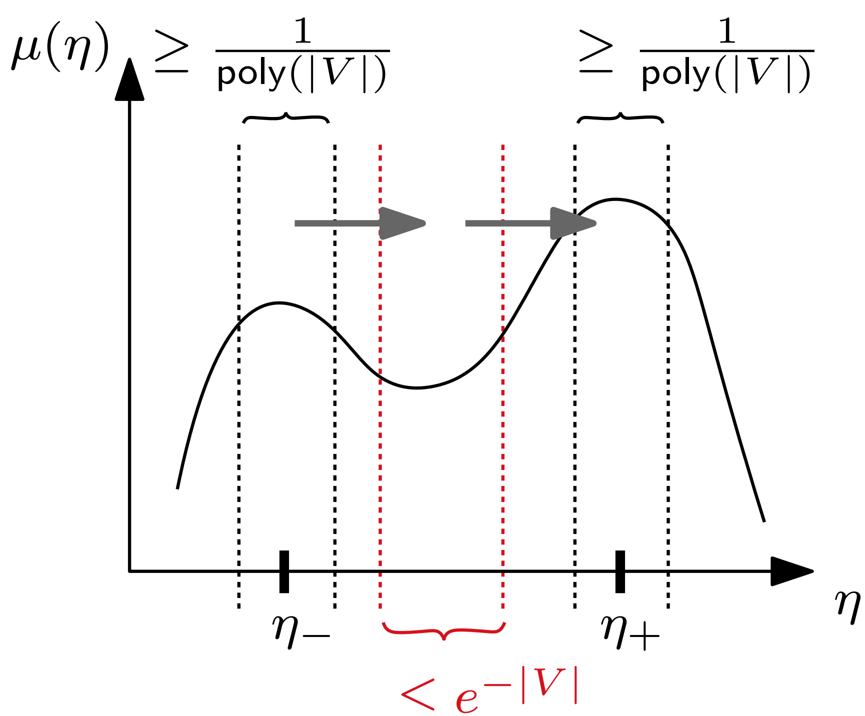

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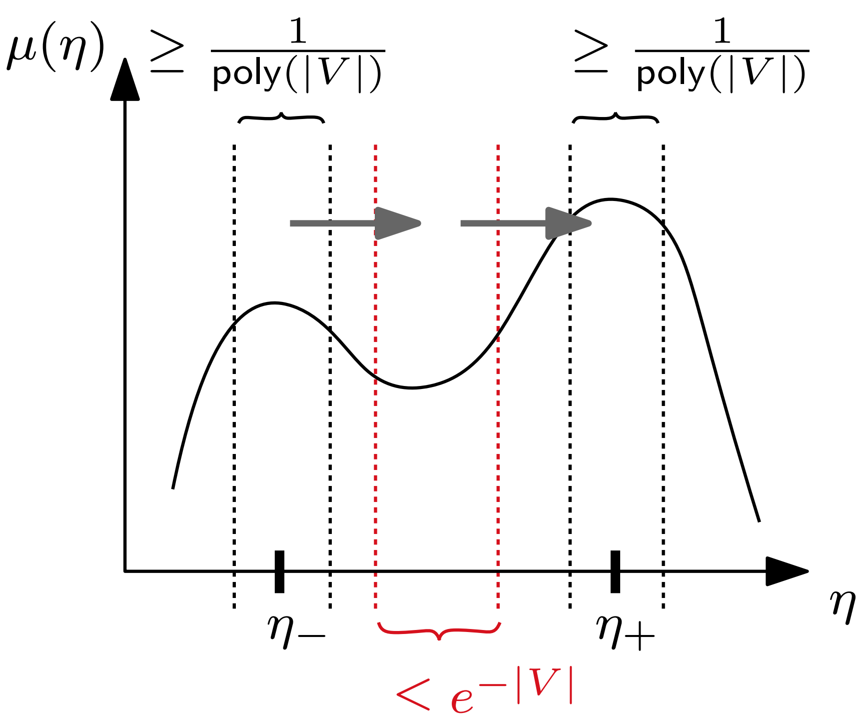
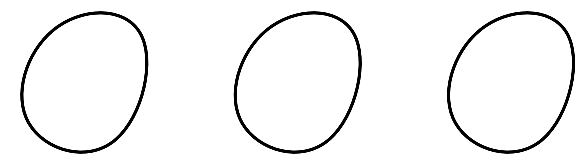


Kawasaki dynamics:

For $H = m \times G$ for large m



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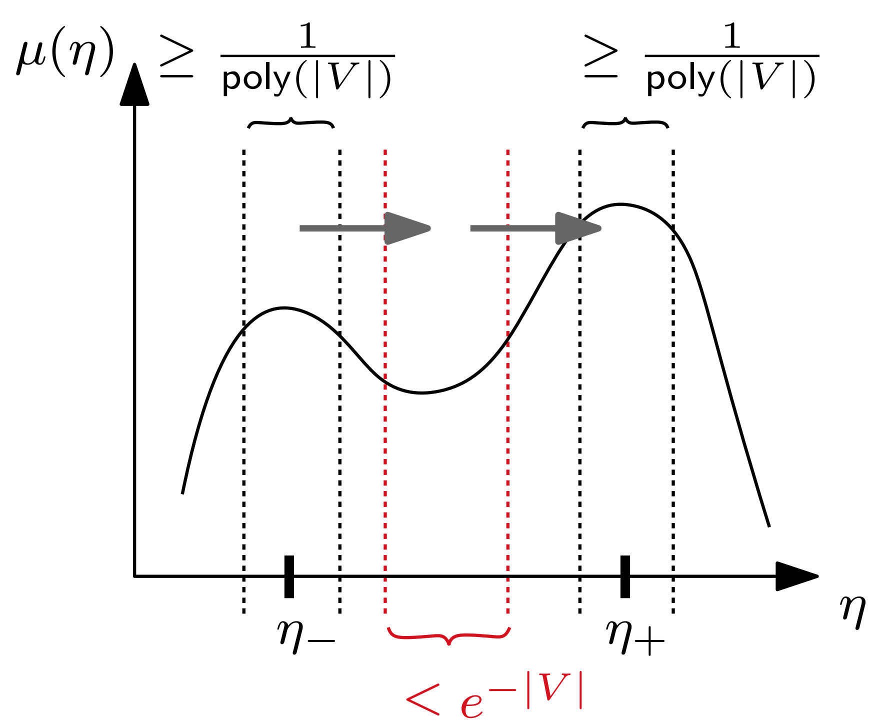

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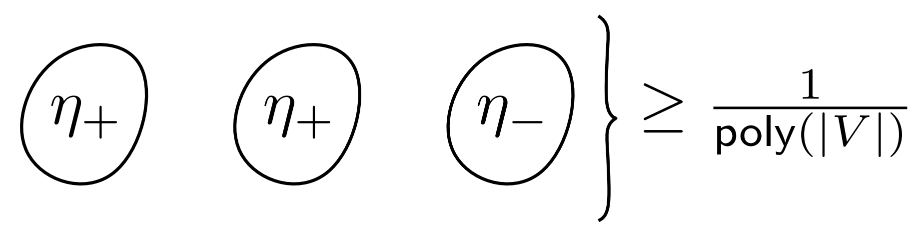

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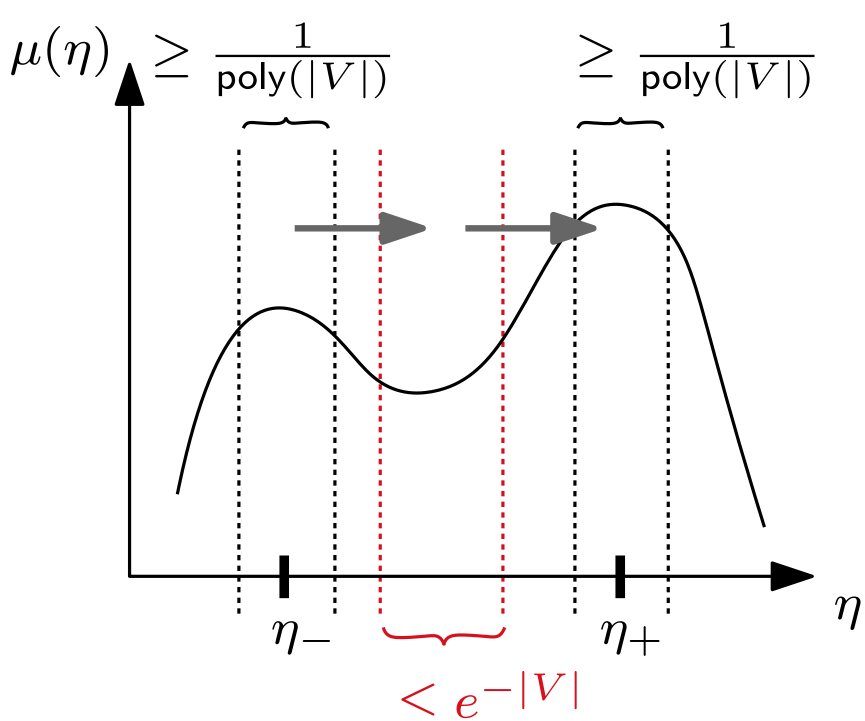

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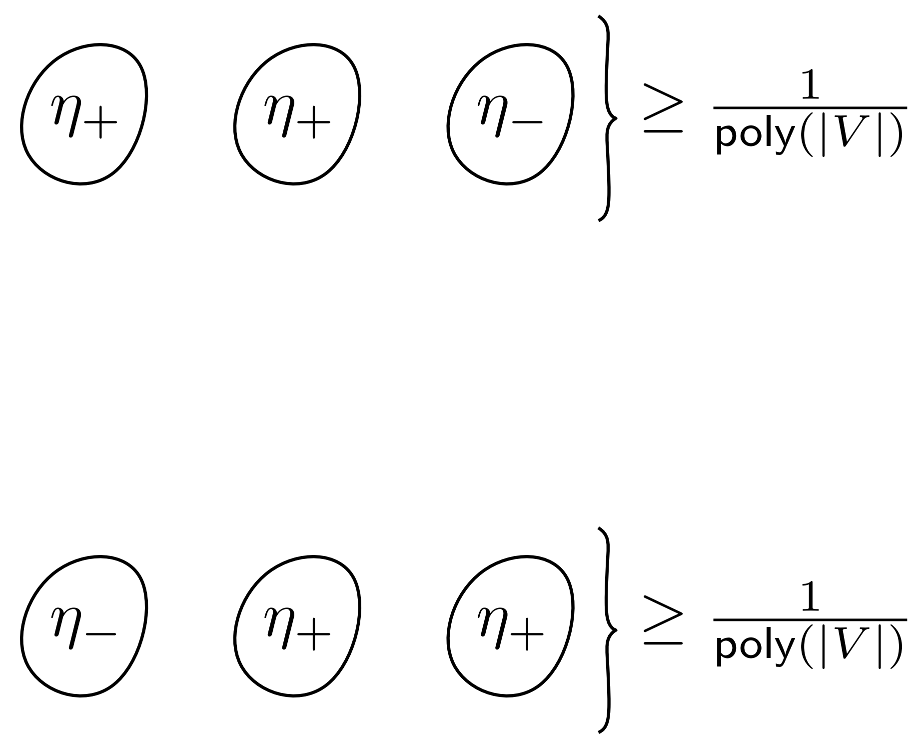
For $H = m \times G$ for large m



Proof Overview: Slow Mixing

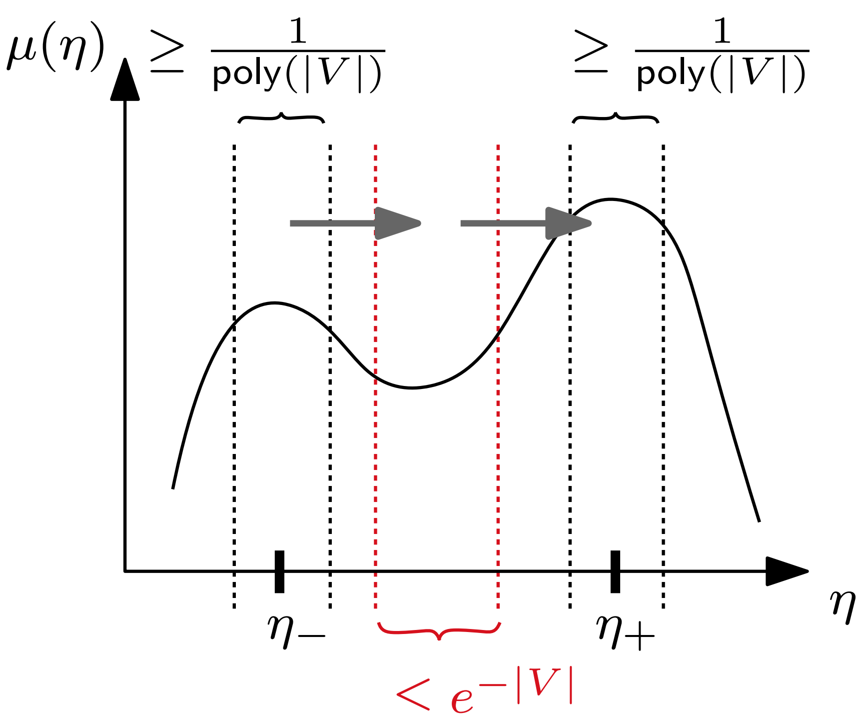
non-uniqueness of ferro. Ising on $\mathbb{T}_\Delta$ $\xrightarrow[\text{[GSVY'16]}]{}$ metastability of ferro. Ising on Δ-regular random graphs

→ slow mixing of Glauber dynamics

→ slow mixing of Kawasaki dynamics

Glauber dynamics:

For $G \sim U(\mathcal{G}_\Delta)$ we have w.h.p.



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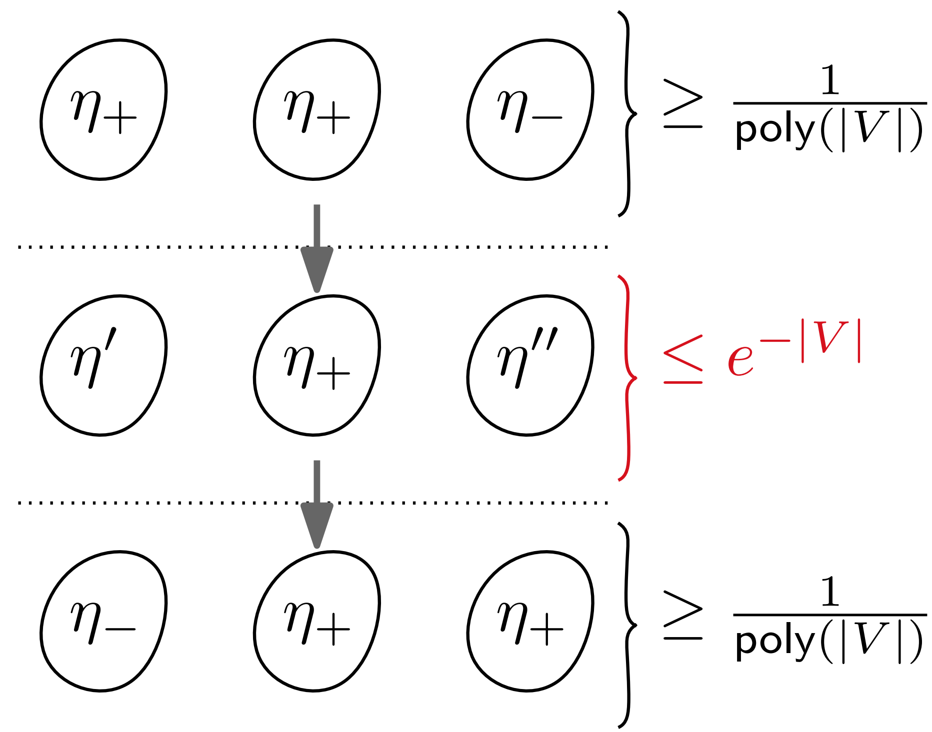

Final: Open Questions, Conjectures and Future Work

Conjectures and Open Questions

Conjecture
If $\beta < \beta_u(\Delta)$ or $\eta \notin [-\eta_a, \eta_a]$ then Kawasaki dynamics have mixing time $O(|V(G)| \cdot \log |V(G)|)$ for all $G \in \mathcal{G}_\Delta$.

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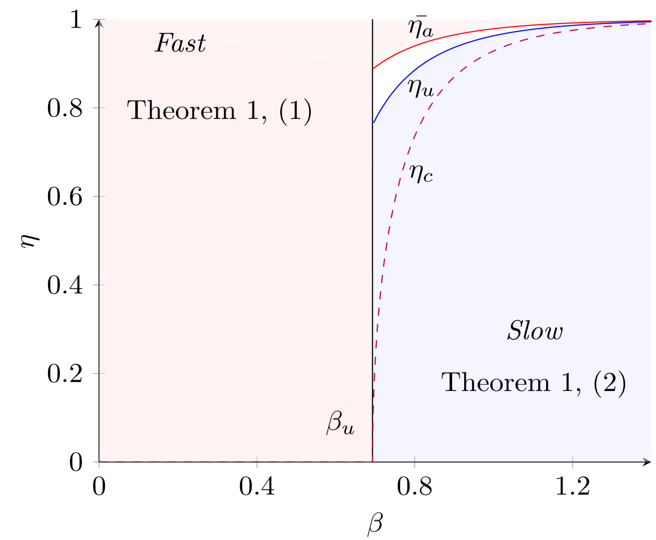
Is $\lambda_a(\Delta, \beta) = \lambda_u(\Delta, \beta)$? (if yes then $\eta_a(\Delta, \beta) = \eta_u(\Delta, \beta)$)



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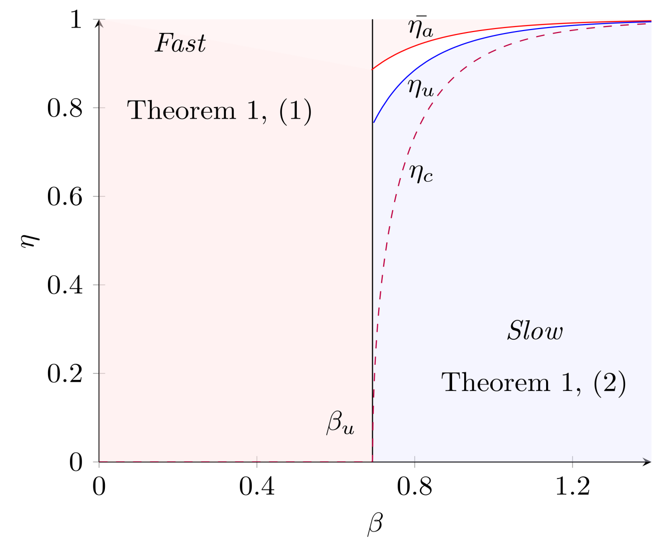

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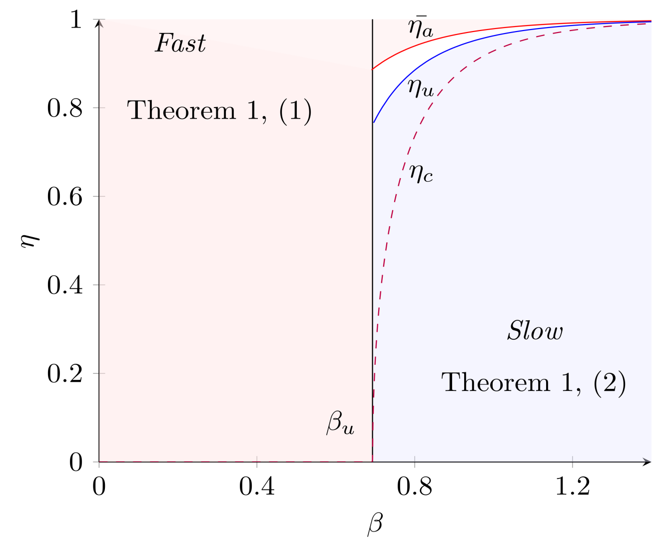

Better Question

Are Kawasaki dynamics rapidly mixing for all $\eta \notin [-\eta_u, \eta_u]$?

Conjectures and Open Questions 2

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More Precise Question

Fix G, $\beta > \beta_u$, $0 < \lambda_0 < 1/\lambda_u$, and, for every "not too big" $S \subset V$, let λ_S be such that

$$\mathbb{E}_{\sigma \sim \mu_{\lambda_S}^{S \mapsto +1}}[\eta(\sigma)] = \mathbb{E}_{\tau \sim \mu_{\lambda_0}}[\eta(\tau)] .$$

Does it hold that $\lambda \mapsto Z^{S \mapsto +1}(\lambda)$ is zero-free in a neighborhood of λ_S?

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Thank you!

Backup

Side Result: Ferromagnetic Ising Model

Reminder:
Slow mixing of Glauber dynamics is only known for $\beta > \beta_u(\Delta)$ and $\lambda = 1$.

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For all $\beta > \beta_u(\Delta)$ and $\lambda \in \left(\frac{1}{\lambda_u(\Delta,\beta)}, \lambda_u(\Delta, \beta)\right)$, there is a sequence $G_n \in \mathcal{G}_\Delta$ with $|V(G_n)| \to \infty$ such that the mixing time of Glauber dynamics is **exponential** in $|V(G_n)|$.

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