# The independence polynomial on recursive graphs – the dynamical perspective

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University of Amsterdam

Workshop on the CAP aspects of Partition Functions
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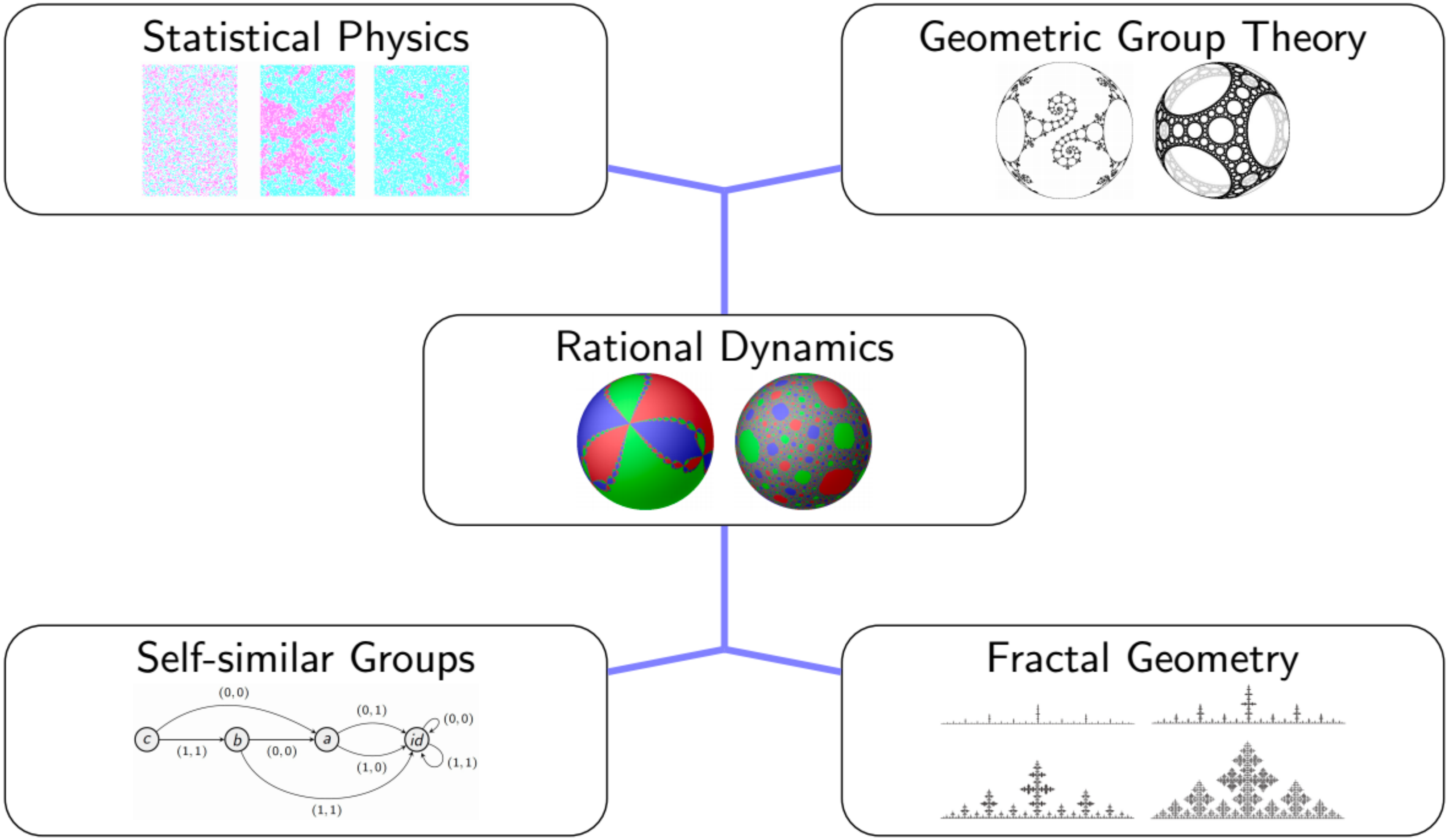
"Understand relations between different measures of complexity of dynamical systems, fractal sets, graphs, and groups."




Pictures courtesy of C. Bishop, C. McMullen, and P. Winkler

# CODAG: Complexity in Dynamical systems, Algebra, and Geometry

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## Main Characters

- Independence polynomial and its zeroes
- Free energy and phase transitions
- Recursive graphs
- Rational dynamical systems
- Invariant variety
- . . .

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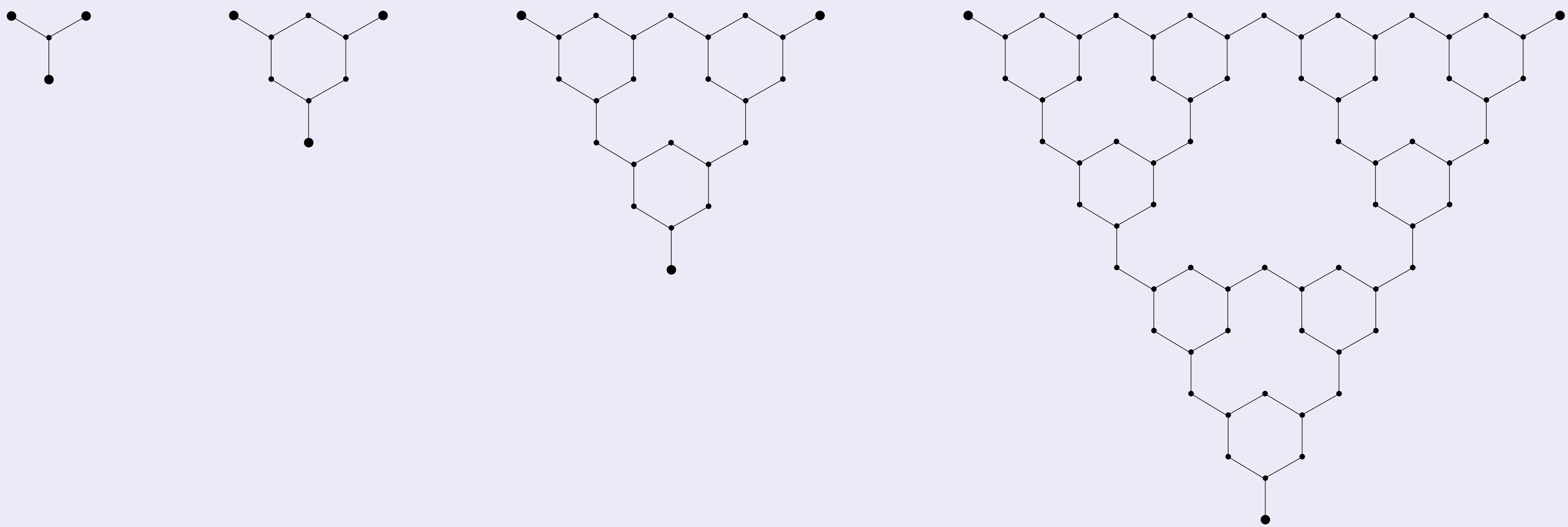

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- If the recursion operator $R$ is non-degenerate and expanding then **zeros of the independence poly's $Z_{G_n}(\lambda)$ avoid a nbhd of $\mathbb{R}_+$**.

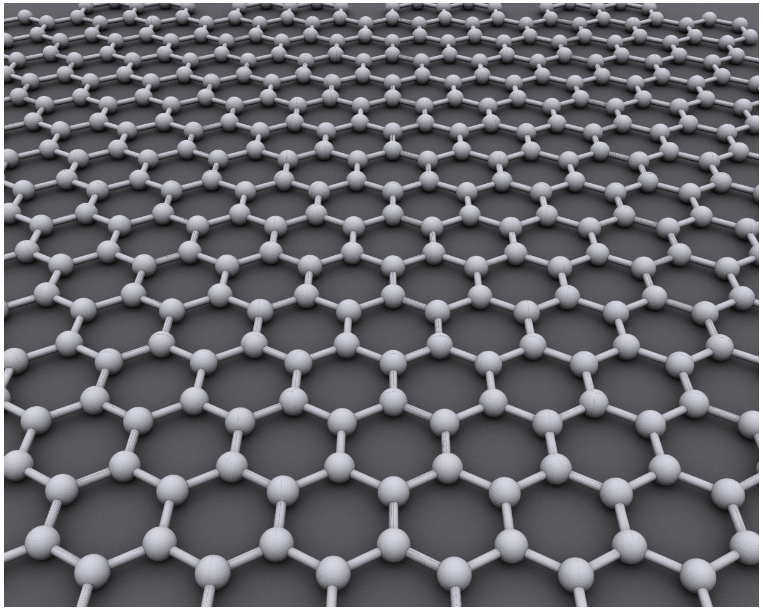
**Corollary:** The limiting free energy is well-defined and real analytic on all of $\mathbb{R}_+$: there are **no phase transitions**.

Prologue:

# Partition Functions and Phase Transitions

# Motivation from **Statistical Physics**

The **Hard-Core** (**Neighbour Exclusion**) **model** on a (possibly infinite) graph $G$ represents the behavior of large particles at the vertices of $G$ which excludes the presence of other particles at the adjacent sites.



Gas molecules absorbed in the dual of a graphene lattice

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A **(spin) configuration on** $G$ is a vertex assignment $\sigma : V(G) \to \{0, 1\}$.

The **weight** of $\sigma$ is

$$e^{-H(\sigma,\lambda)} = \begin{cases} \lambda^{\#\sigma^{-1}(1)}, & \sigma \text{ is independent} \\ 0, & \text{otherwise.} \end{cases}$$

The **partition function** is

$$Z_G(\lambda) := \sum_{\text{ind. } \sigma : V(G) \to \{0,1\}} \lambda^{\#\sigma^{-1}(1)}$$

— the **independence polynomial** of $G$.

# **Phase transitions** (for the Hard-Core model)

Let $\{G_n\}$ be a finite graph sequence approximating a limiting graph $G_\infty$.

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## Ehrenfest classification

The **pressure** (or **limiting free energy per site**) for $\{G_n\}$ is defined as

$$\mathcal{P}(\lambda) := \lim_{n\to\infty} \frac{\log Z_{G_n}(\lambda)}{\# V(G_n)} \qquad (\text{for } \lambda \in \mathbb{R}_+).$$

**Phase transition of order $k$ at $\lambda_0 \in \mathbb{R}_+$** — discontinuity of the $k$-th order derivative of the limiting free energy $\mathcal{P}$ at $\lambda_0$.

# Phase transitions for regular lattices

**Folklore Conjecture**

Let $\{G_n\}$ be a sequence of finite graphs approximating a regular lattice. Then there exists a **unique critical parameter $\lambda_c$** $\in \mathbb{R}_+$ such that

$$\begin{cases} \lambda_0 < \lambda_c & \Rightarrow \quad \text{unique Gibbs measure at } \lambda_0 \\ \lambda_0 > \lambda_c & \Rightarrow \quad \text{multiple Gibbs measures at } \lambda_0. \end{cases}$$

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## [Yang-Lee'1952]

For "good" graph sequences approximating the lattice $\mathbb{Z}^d$:

- **the limiting free energy is well-defined and continuous on** $\mathbb{R}_+$.
- If zeros of the polynomials $Z_{G_n}(\lambda)$ avoid a **complex nbhd** of $\lambda_0 \in \mathbb{R}_+$, then **the limiting free energy is real analytic at $\boldsymbol{\lambda_0}$**.

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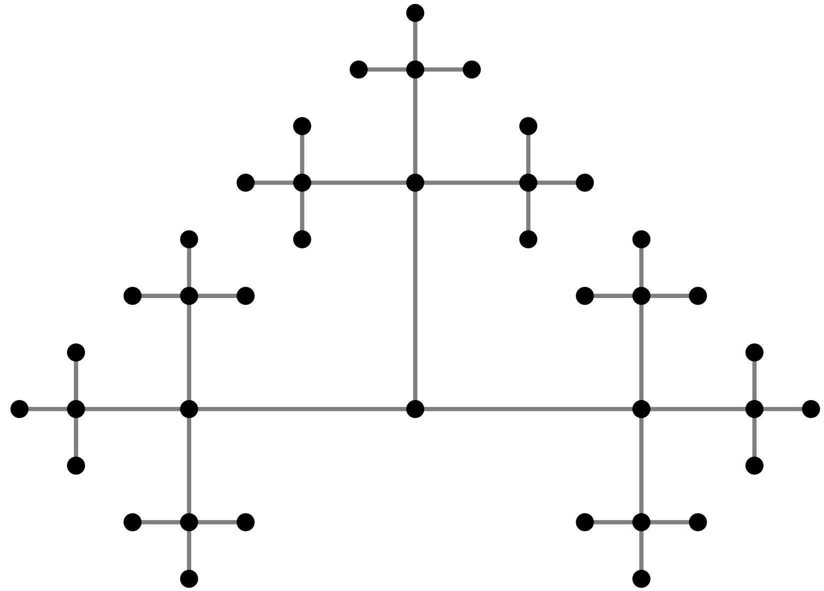
**Regular lattice are hard!**

Chapter I:

# Recursive Graphs

# Examples of recursive graphs I — **Regular rooted trees**

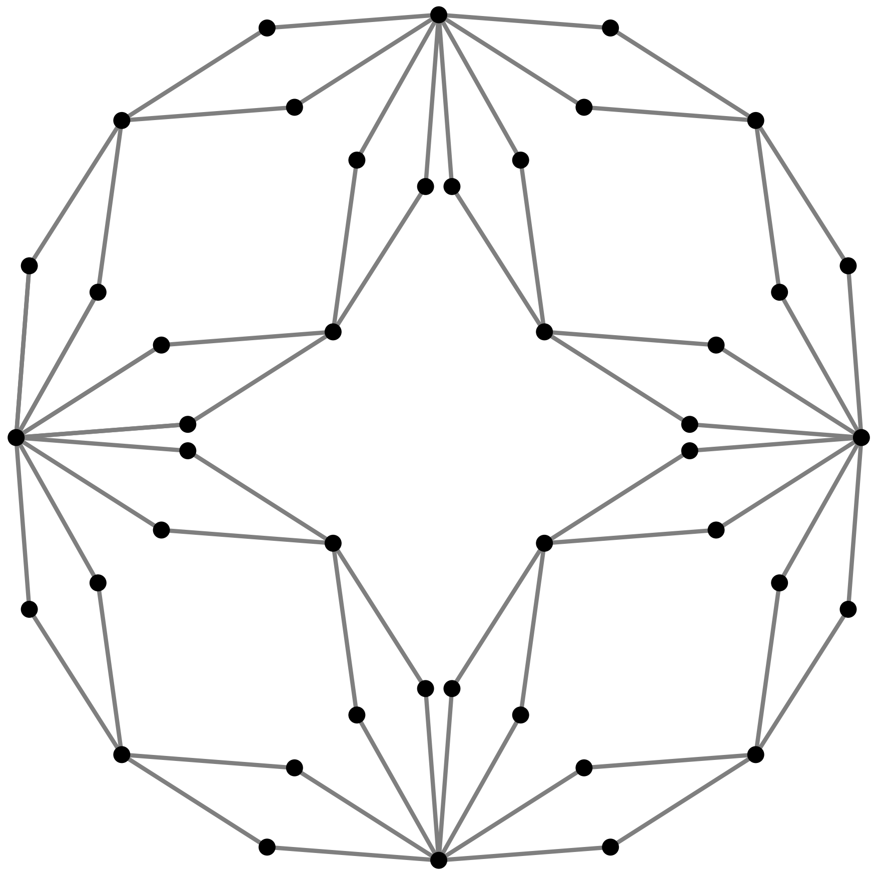


[Rivera-Letelier & Sombra'2019 (talk at the Fields Institute)]

For the **Hard-Core model** on ***d*-ary rooted trees** there is a **unique phase transition** (of infinite order).

Zeros accumulate at a unique parameter in $\mathbb{R}_+$ given by $\lambda(d) = \frac{d^d}{(d+1)^{d-1}}$.

# Examples of recursive graphs II — **Hierarchical lattices**



[Bleher-Lyubich-Roeder'2010, Chio-Roeder'2021]

For the **Ising model** on **diamond hierarchical lattices** there is a **unique phase transition**.

# Recursive graphs

$\mathcal{G}_k = \{\text{finite graphs } G \text{ with } k \text{ vertices labeled } 1, \ldots, k\}$

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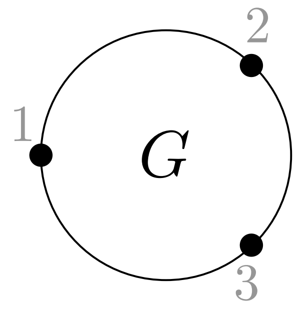

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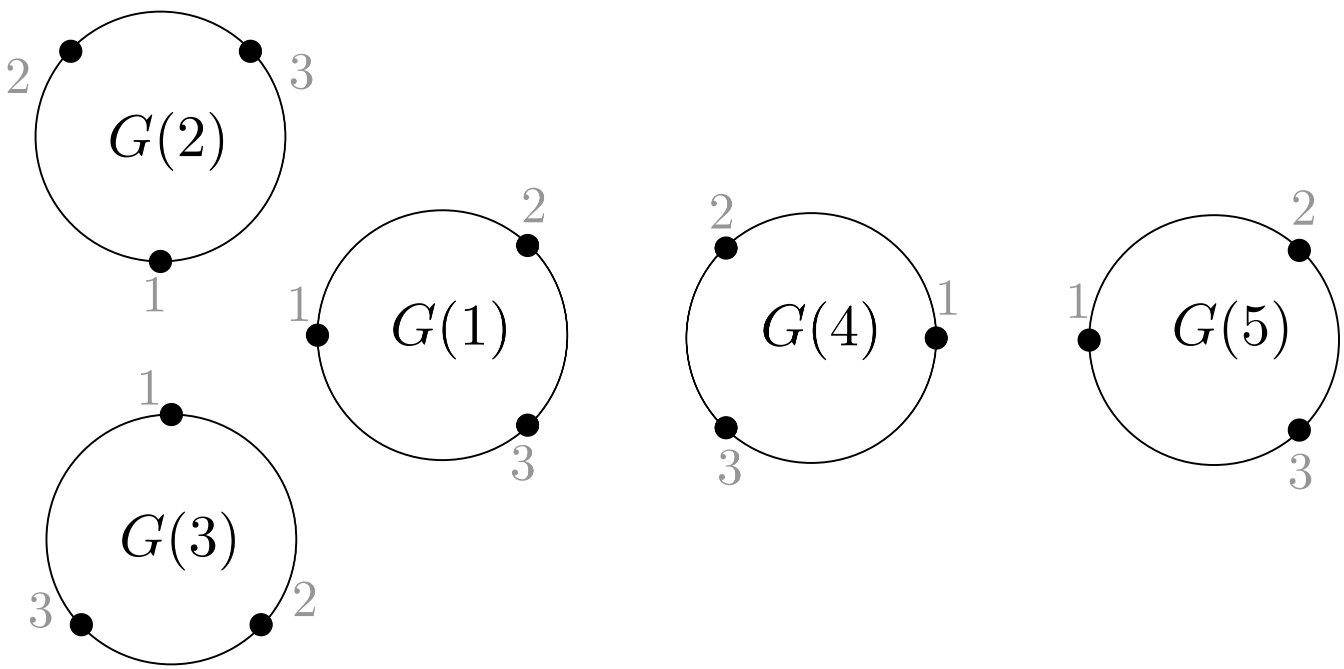

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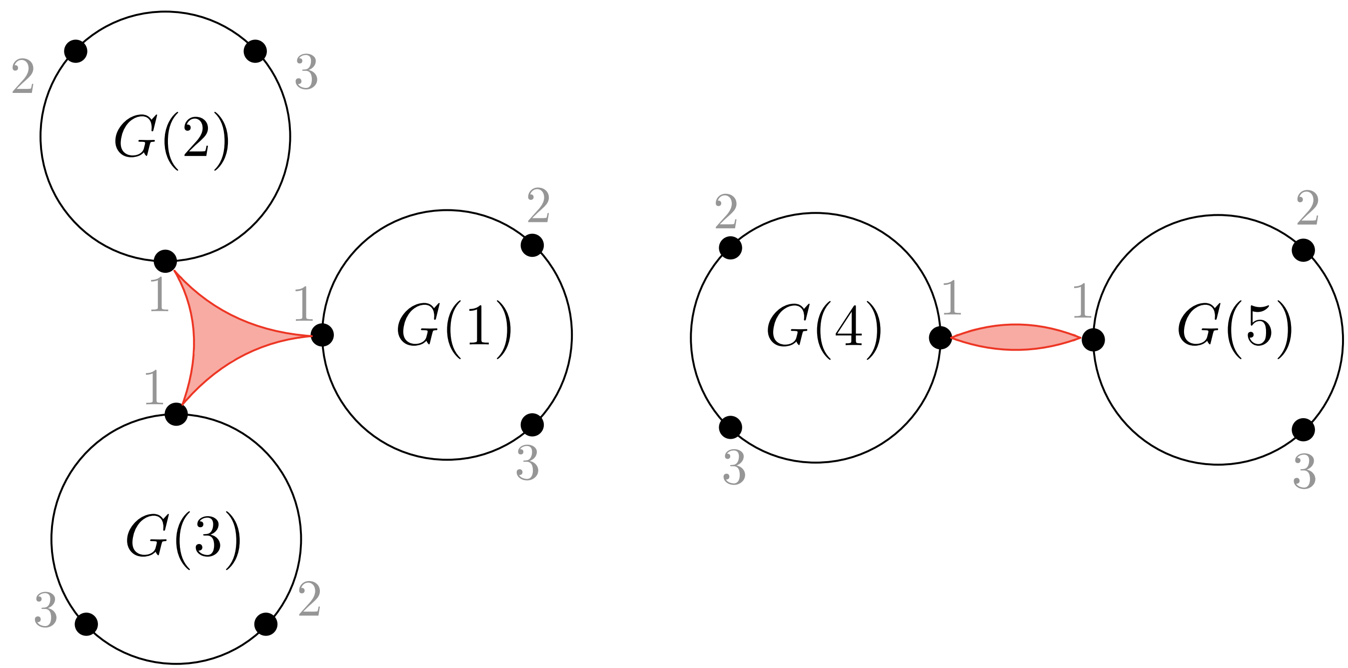

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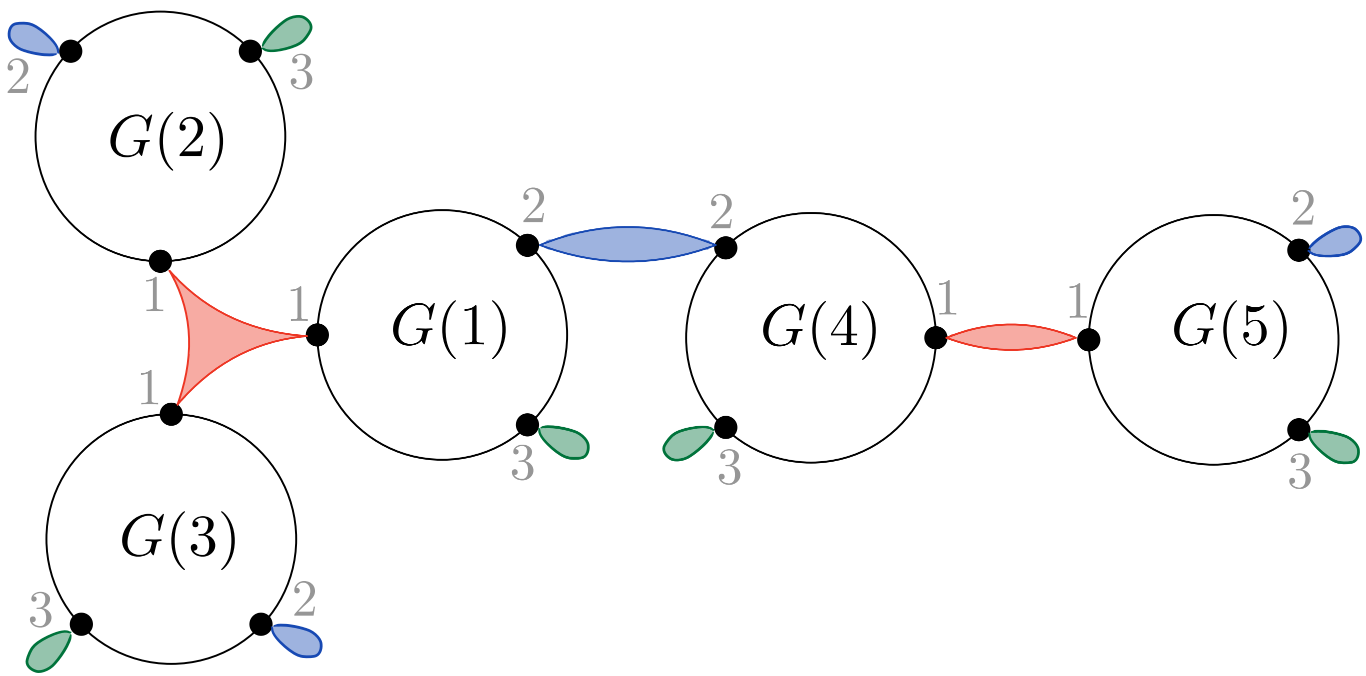

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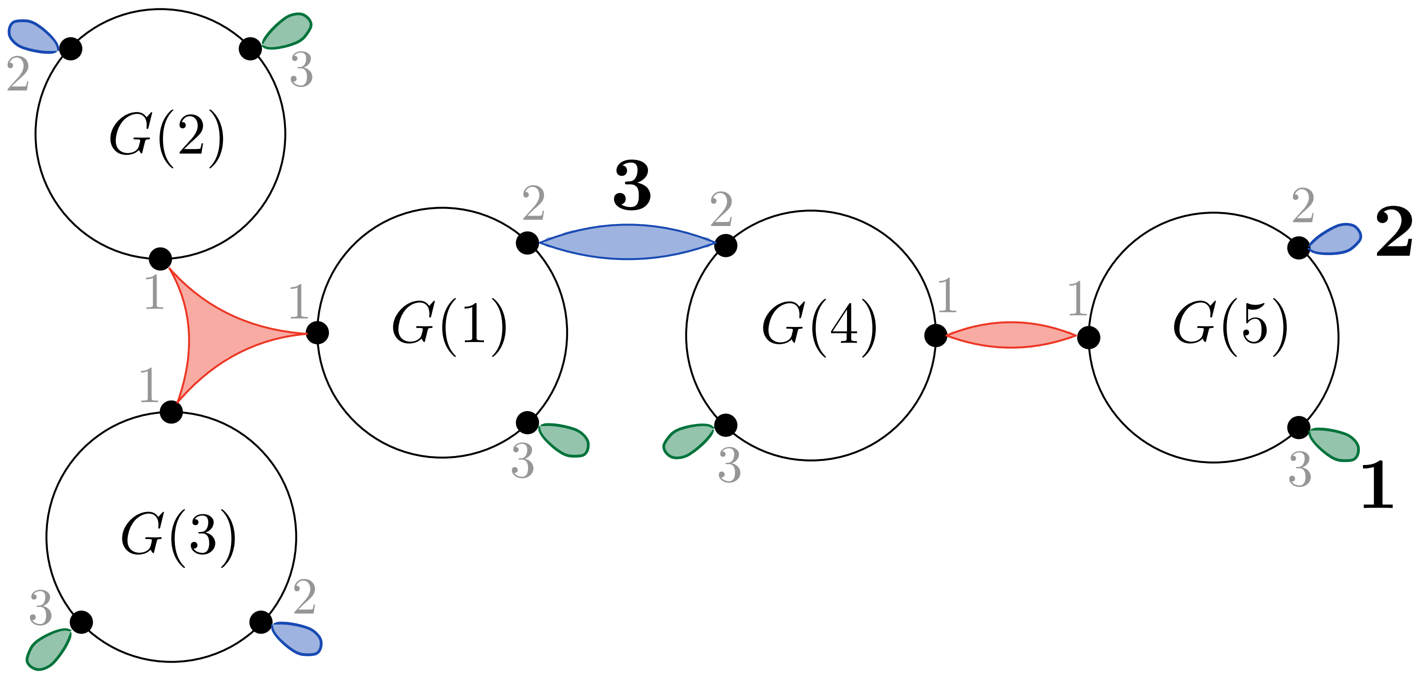

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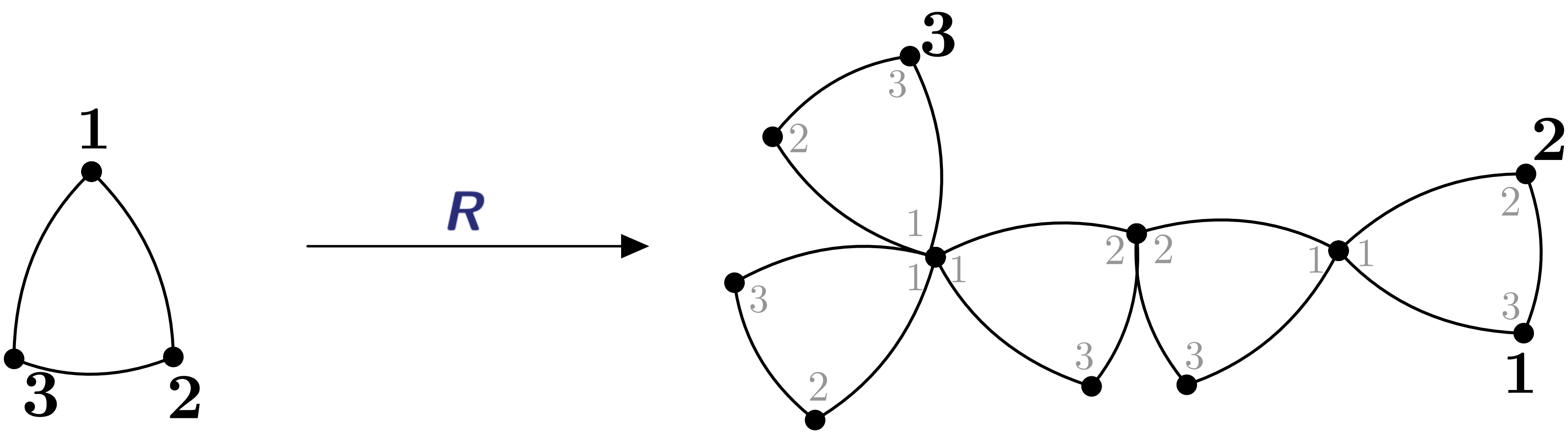
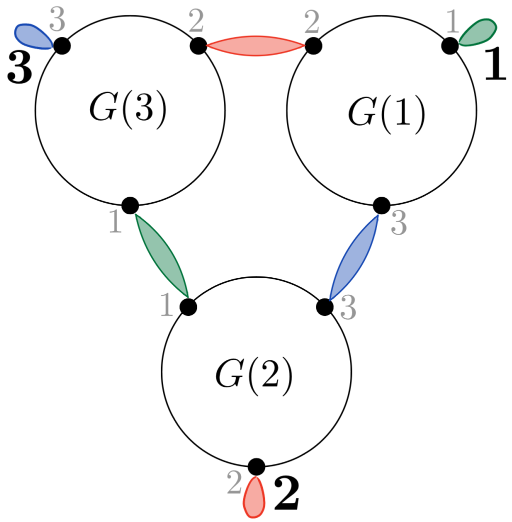
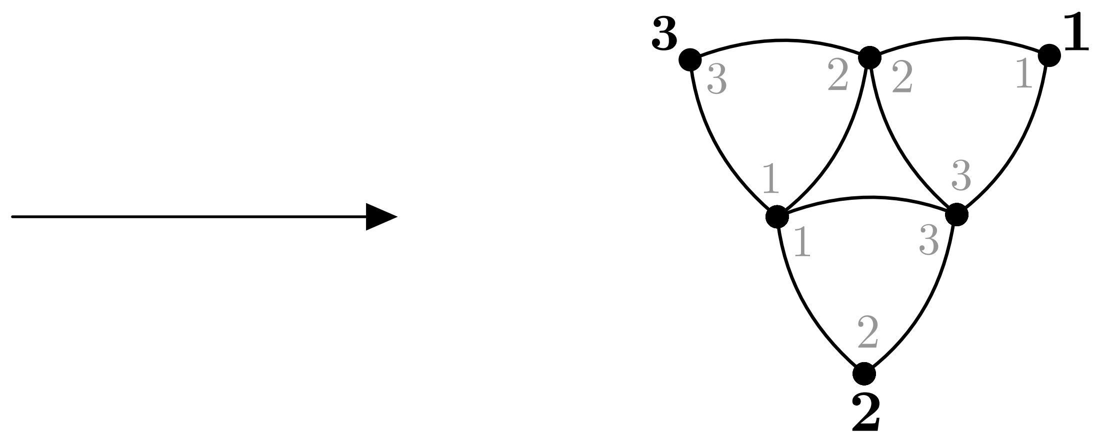
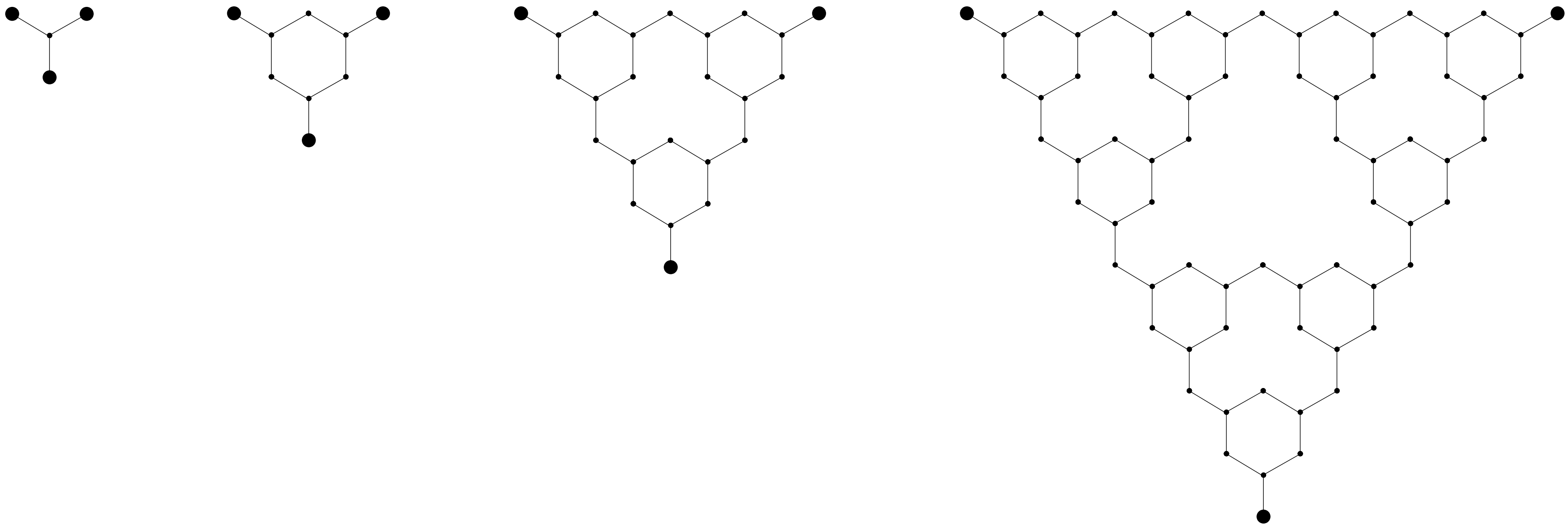


# Examples of recursions — Sierpiński gasket recursion



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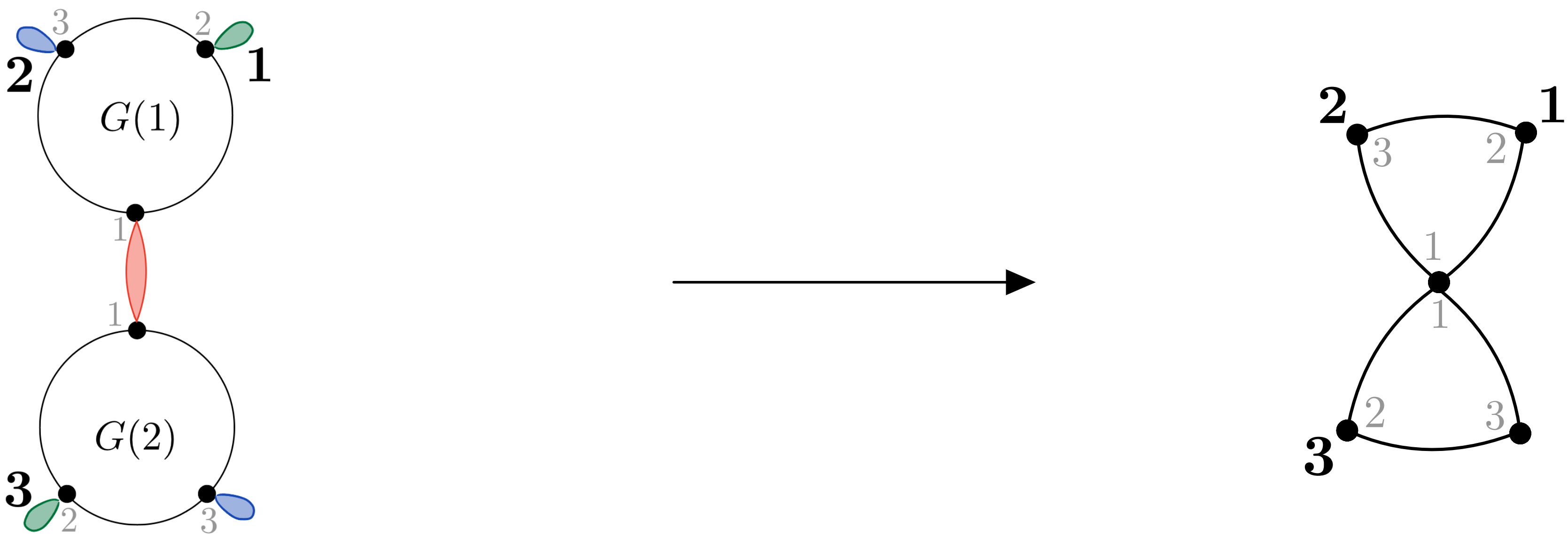


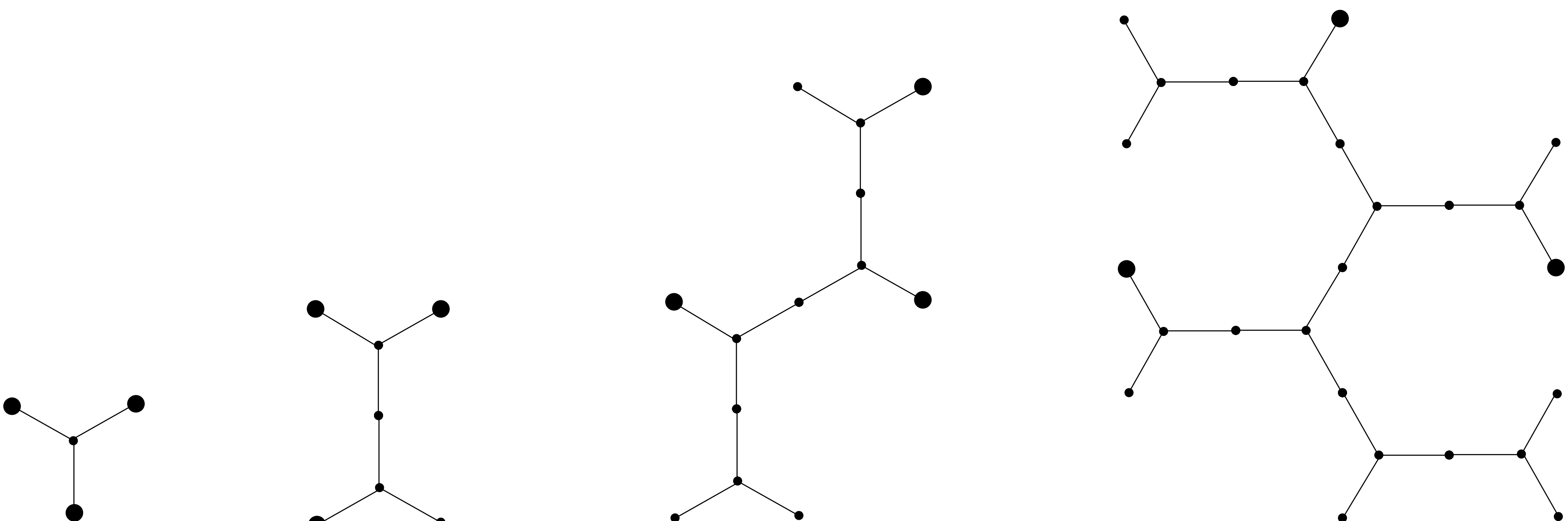

# Non-degenerate and expanding recursions

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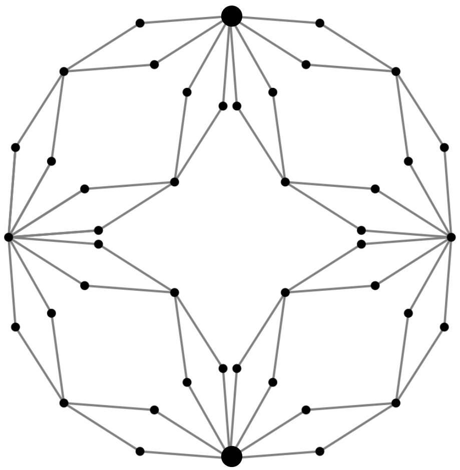
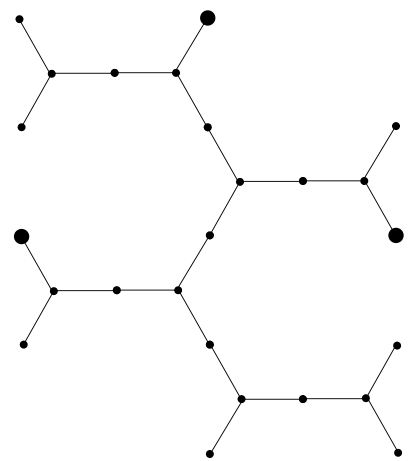
Starting graph $G_0 \in \mathcal{G}_k$ $\rightsquigarrow$ recursive graph sequence $\{G_n = R^n(G_0)\}$

$R$ is **non-degenerate** if for some (and thus for all) connected $G_0$ the vertex degrees of $G_n$ are uniformly bounded (in $n$).





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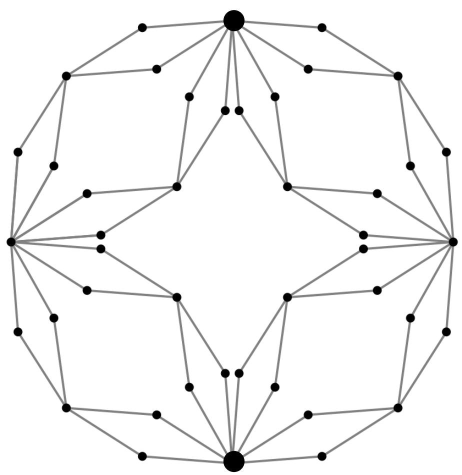
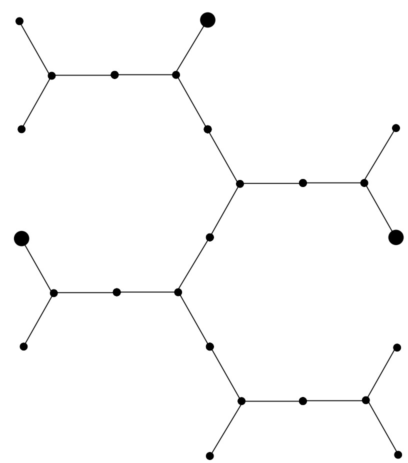
$R$ is **expanding** if for some (and thus for all) connected $G_0$ the distance between vertices labeled $\ell, \ell'$ in $G_n$ diverges to $\infty$ as $n \to \infty$ for all $\ell \neq \ell' \in \{1, \ldots, k\}$.





Chapter II:

# Dynamical System

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For $\overline{x} = (x_1, \ldots, x_k) \in \{0,1\}^k$ — "an assignment on $L(G)$", we set

$$\boldsymbol{Z_G^{\overline{x}}(\lambda) := \sum_{\substack{\text{ind. } \sigma: V(G) \to \{0,1\} \\ \sigma \sim \overline{x} \text{ on } L(G)}} \lambda^{\#\sigma^{-1}(1)}}.$$

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Then

$$Z_G(\lambda) = \sum_{\overline{x} \in \{0,1\}^k} Z_G^{\overline{x}}(\lambda).$$

We have a natural **coordinate map** $\boldsymbol{\phi_\lambda : \mathcal{G}_k \to \mathbb{C}^{2^k}}$

$$G \mapsto \left( Z_G^{(0,\ldots,0)}(\lambda), \ldots, Z_G^{(1,\ldots,1)}(\lambda) \right).$$

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## Claim

Coordinates of $\phi_\lambda(R(G))$ can be expressed as homogeneous polynomials in coordinates of $\phi_\lambda(G)$.

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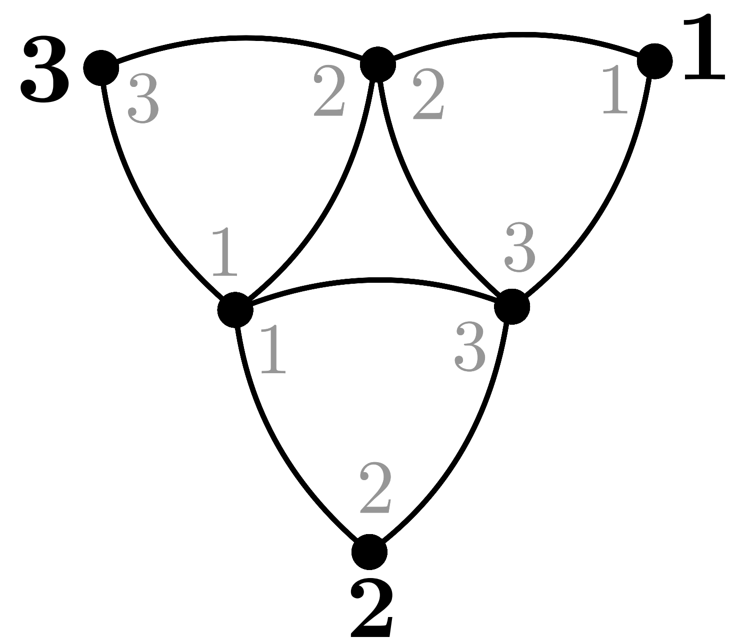

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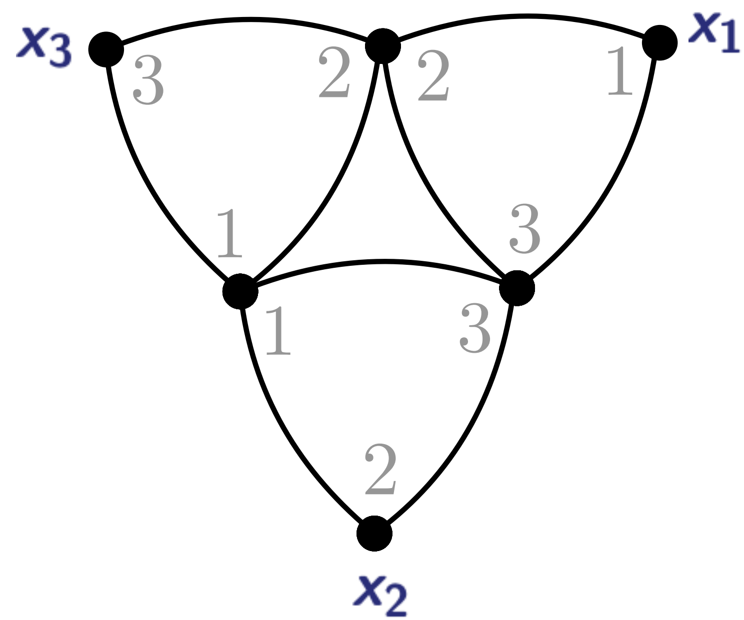


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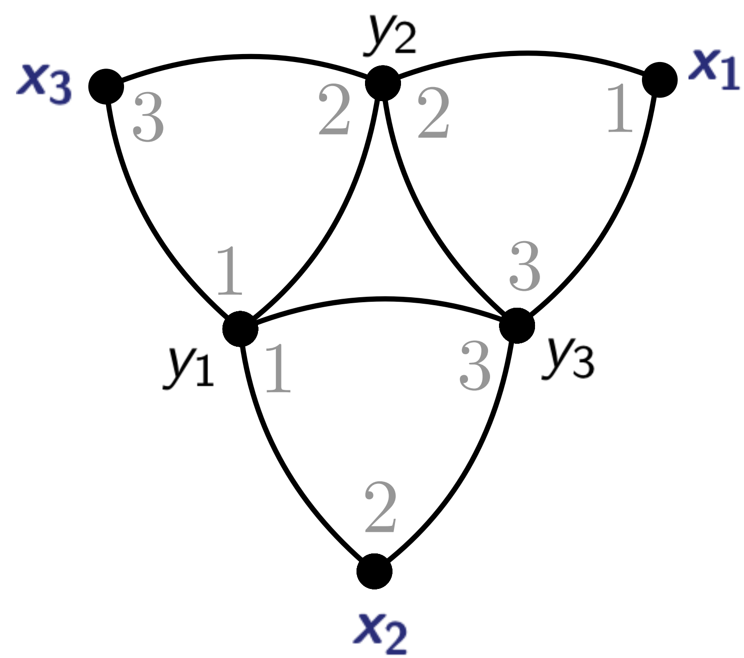


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$$(x_1, x_2, x_3)' = \sum_{(y_1,y_2,y_3)\in\{0,1\}^3} \frac{(x_1, y_2, y_3)\cdot(y_1, x_2, y_3)\cdot(y_1, y_2, x_3)}{\lambda^{y_1+y_2+y_3}}.$$

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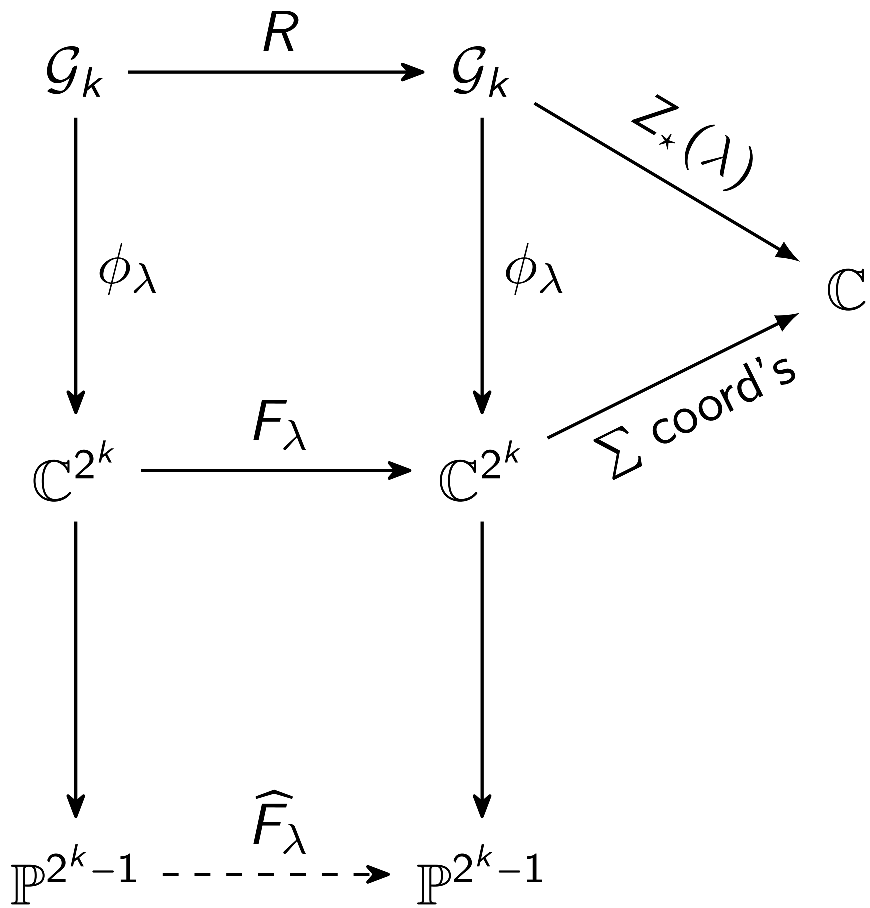

Chapter III:

# Invariant Variety

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## Lemma

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for all $\overline{x}, \overline{y} \in \{0,1\}^k$ with $\overline{x} + \overline{y} \in \{0,1\}^k$ (i.e., with disjoint support).

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Then **$\dim(\mathcal{M}) = k$** and **$\mathcal{M}$ is** (forward) **invariant under $\widehat{F}_\lambda$**.

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for all $\overline{x}, \overline{y} \in \{0,1\}^k$ with $\overline{x} + \overline{y} \in \{0,1\}^k$ (i.e., with disjoint support).

Then **$\dim(\mathcal{M}) = k$** and **$\mathcal{M}$ is** (forward) **invariant under $\widehat{F}_\lambda$**.

**Note:** $\mathcal{M}$ depends ONLY on $k$! and not on the recursion data $(m, H, \Phi)$!

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- Can be proven directly using the formula for $F_\lambda$.
- Alternatively, one can use a *probability interpretation* of $\mathcal{M}$.

# Probability interpretation of the invariant variety $\mathcal{M}$

Let $\tau : L \to \{0,1\}$ be an assignment on $L \subset L(G)$. Set

$$\mathbb{P}_G[\tau] := \sum_{\text{ind. } \sigma \text{ with } \sigma|_L = \tau} \lambda^{\#\sigma^{-1}(1)} / Z_G(\lambda) = \sum_{\overline{x} \in \{0,1\}^k,\ \tau \sim \overline{x} \text{ on } L} Z_G^{\overline{x}}(\lambda) / Z_G(\lambda)$$

— the probability that vertices of $L$ get the assignment $\tau$ (for $\lambda \in \mathbb{R}^+$).

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## Claim

The variety $\mathcal{M} \subset \mathbb{P}^{2^k - 1}$ is defined by the following equations:

$$\mathbb{P}_G[\tau \cap \tau'] = \mathbb{P}_G[\tau] \cdot \mathbb{P}_G[\tau']$$

for any assignments $\tau, \tau'$ on disjoint subsets of $L(G)$.

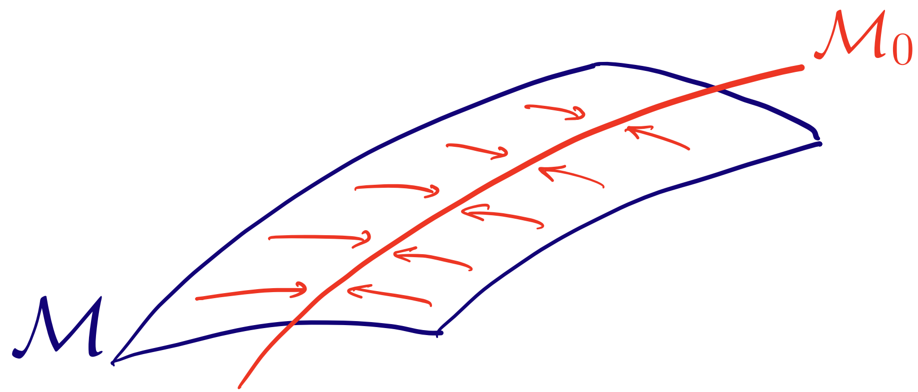
In other words: **no correlation between different labeled vertices**!

# Dynamics **on** the invariant variety

**Lemma**

If $R$ is **non-degenerate**, then $\mathcal{M}$ **is eventually periodic**:

$$\mathcal{M} \xrightarrow{\widehat{F}_\lambda^t} \mathcal{M}_0 \circlearrowleft \mathrm{id} \qquad \text{for some iterate } t \geq 1.$$



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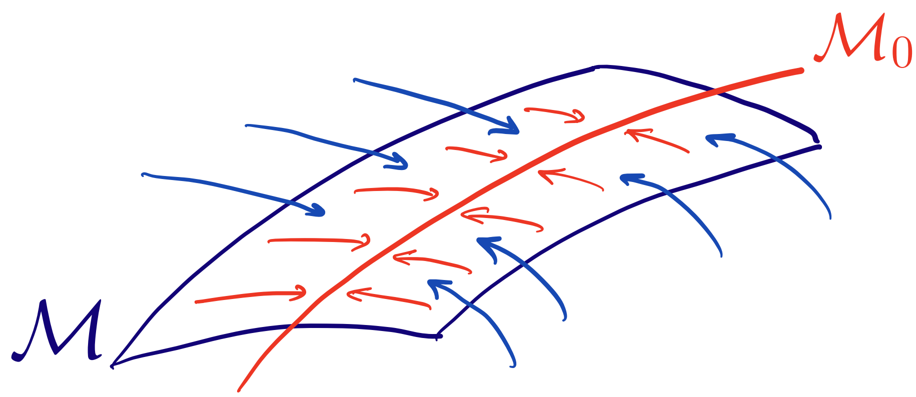
# Dynamics **near** the invariant variety

**Theorem**

If $R$ is **expanding**, then $\mathcal{M}$ **is transversally superattracting**: $\exists C >, \epsilon_0 > 0$ s.t.

$$|\mathbb{P}_G(\tau : \tau') - \mathbb{P}_G(\tau)| < \epsilon < \epsilon_0 \quad \Rightarrow \quad |\mathbb{P}_{R(G)}(\sigma : \sigma') - \mathbb{P}_{R(G)}(\sigma)| < C \cdot \epsilon^2.$$

$\forall \tau, \tau' : L(G) \to \{0,1\}$, $\text{supp}(\tau) \cap \text{supp}(\tau') = \varnothing$ $\quad \forall \sigma, \sigma' : L(R(G)) \to \{0,1\}$, $\text{supp}(\sigma) \cap \text{supp}(\sigma') = \varnothing$



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**Corollary**

Suppose $R$ is **non-degenerate** and **expanding**. Then

- the **spectrum of $J := \mathrm{Jac}_{\widehat{F}_\lambda}(\xi_0)$** at any $\xi_0 \in \mathcal{M}_0$ **is $\{0, 1\}$**;
- $\mu_J(1) = \dim_J(E_1) = \dim(\mathcal{M}_0)$;
- $\mu_J(0) = \dim_{J^t}(E_0) = 2^k - \dim(\mathcal{M}_0) - 1$.

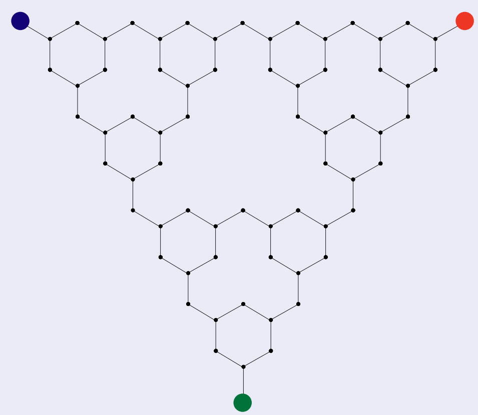
# Dynamics when starting with **physical values ($\lambda \in \mathbb{R}_+$)**

## Theorem

Suppose $R$ is **non-degenerate** and **expanding**, and let $G_n = R^n(G_0)$. Then correlations between labeled vertices in $G_n$ decay exponentially fast:

$$|\mathbb{P}_{G_n}(\tau : \tau') - \mathbb{P}_{G_n}(\tau)| \to 0 \quad \text{as} \quad n \to \infty$$

for any assignments $\tau, \tau'$ on disjoint subsets of $L(G_n)$ (and $\lambda \in \mathbb{R}_+$).



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In other words:

For $\lambda \in \mathbb{R}_+$, if we start to iterate $\widehat{F_\lambda}$ with $[\phi_\lambda(G_0)] \in \mathbb{P}^{2^k}$ for $G_0 \in \mathcal{G}_k$, then

$$\widehat{F_\lambda}^n([\phi_\lambda(G_0)]) = [\phi_\lambda(G_n)] \to \mathcal{M} \quad \text{as} \quad n \to \infty.$$

Chapter IV:

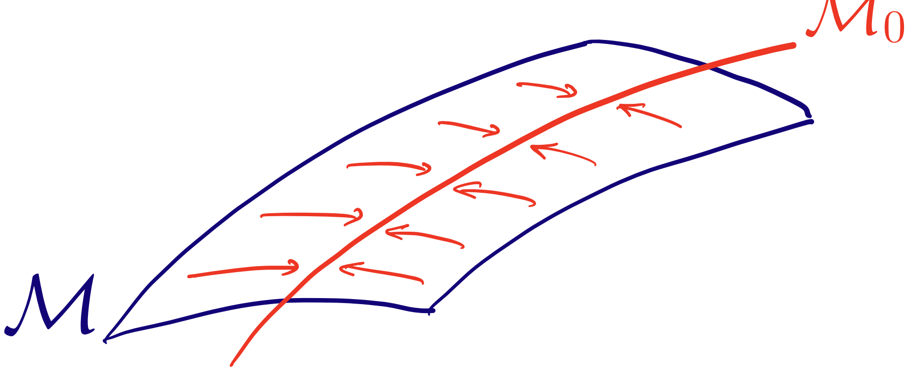
# And they all meet together. . .

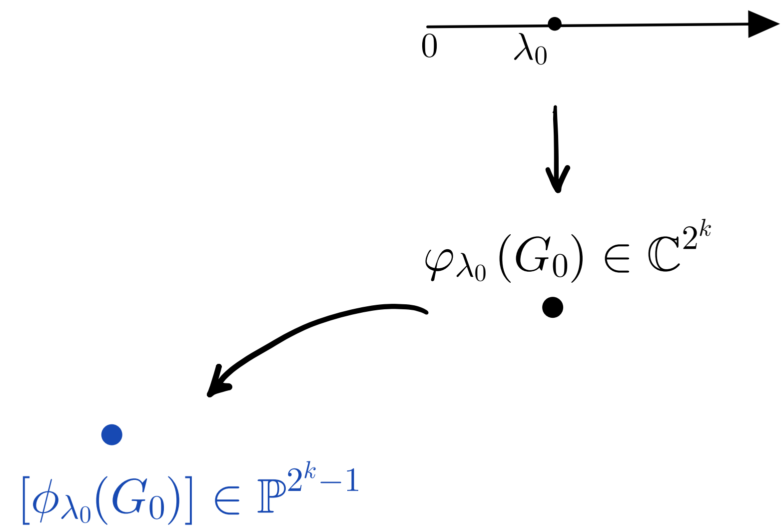
# No Phase Transitions!

## Theorem [H.-Peters]

Suppose $R$ is a **non-degenerate** and **expanding** graph recursion operator, and let $\{G_n = R^n(G_0)\}$ be a recursive graph sequence.

Then **zeros of $Z_{G_n}(\lambda), n \in \mathbb{N}_0$, avoid a uniform neighborhood of $\mathbb{R}_+$.**



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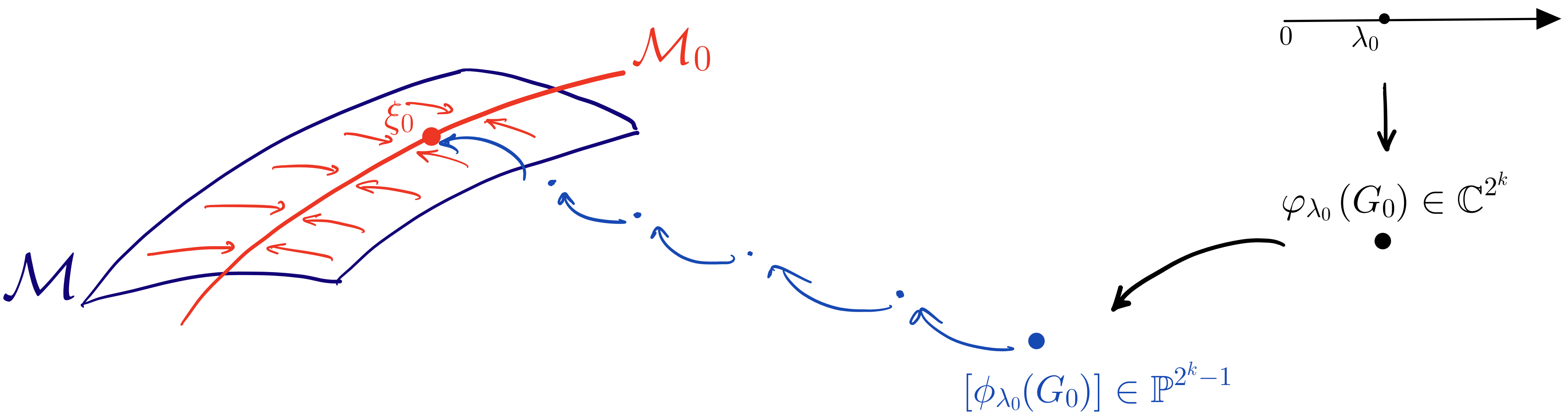

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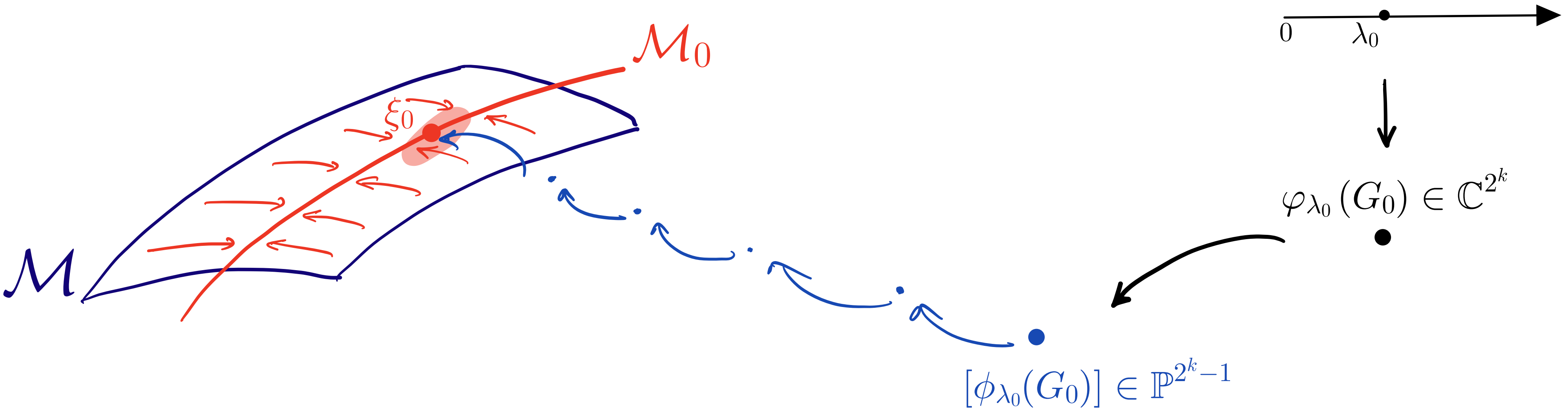

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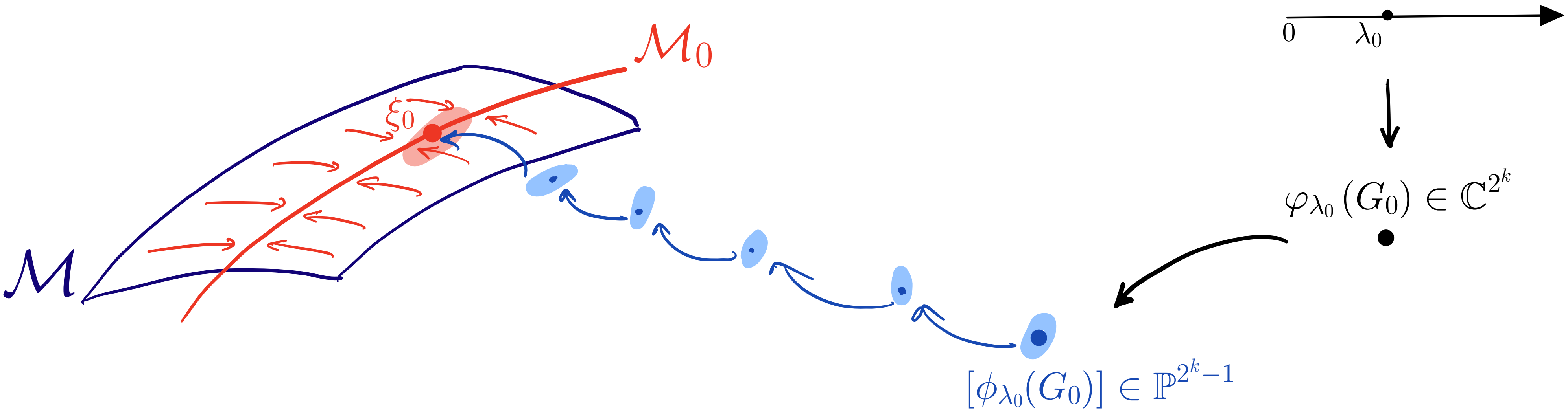

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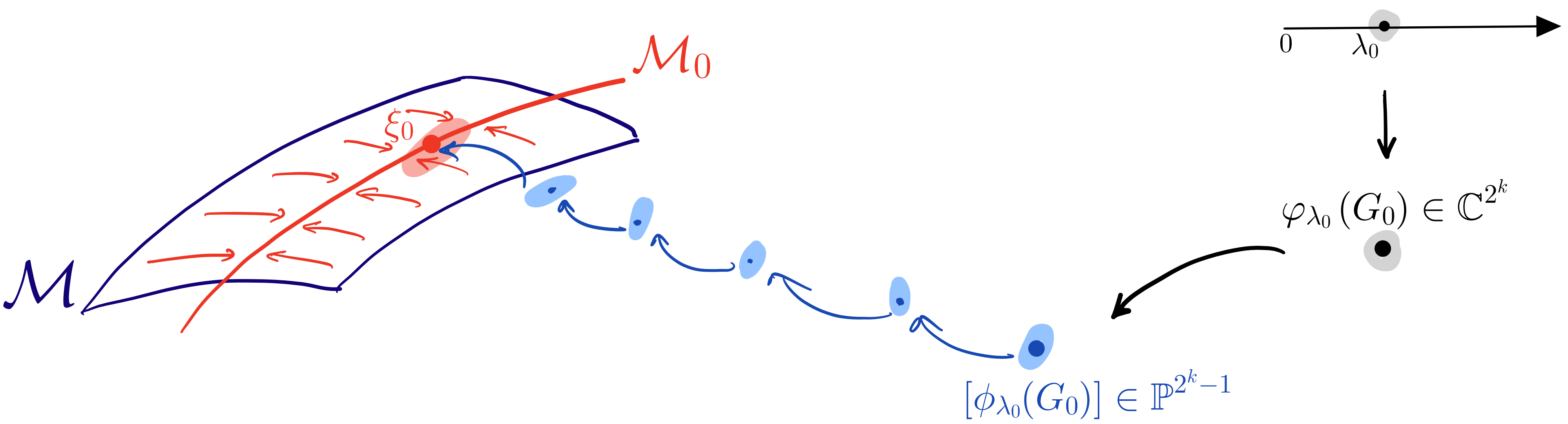

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**Corollary**

The limiting free energy per site is real analytic on all of $\mathbb{R}_+$, that is, there are **no phase transitions**.

# Boundedness of Zeros!

**Theorem [H.-Peters]**

Suppose $R$ is a **non-degenerate** and **expanding** graph recursion operator, and let $\{G_n = R^n(G_0)\}$ be a recursive graph sequence with $\boldsymbol{G_0 = (k+1)}$**–star**.

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Then **zeros of $\boldsymbol{Z_{G_n}(\lambda), n \in \mathbb{N}_0}$, are uniformly bounded**
(and thus avoid a cone around $\mathbb{R}_+$).

Epilogue:

**That is just the beginning!**

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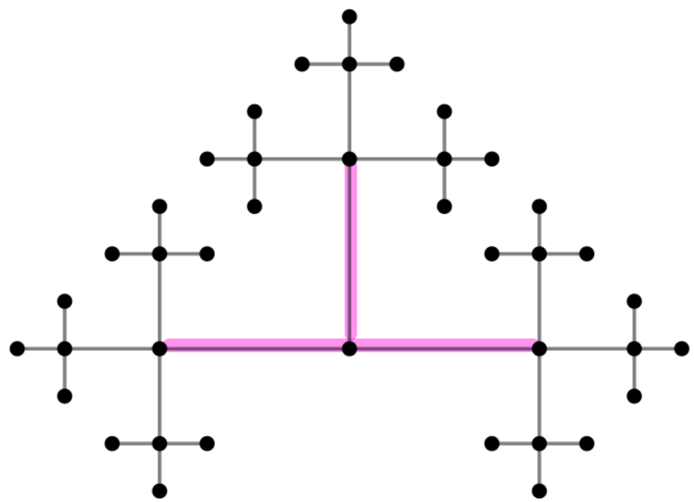

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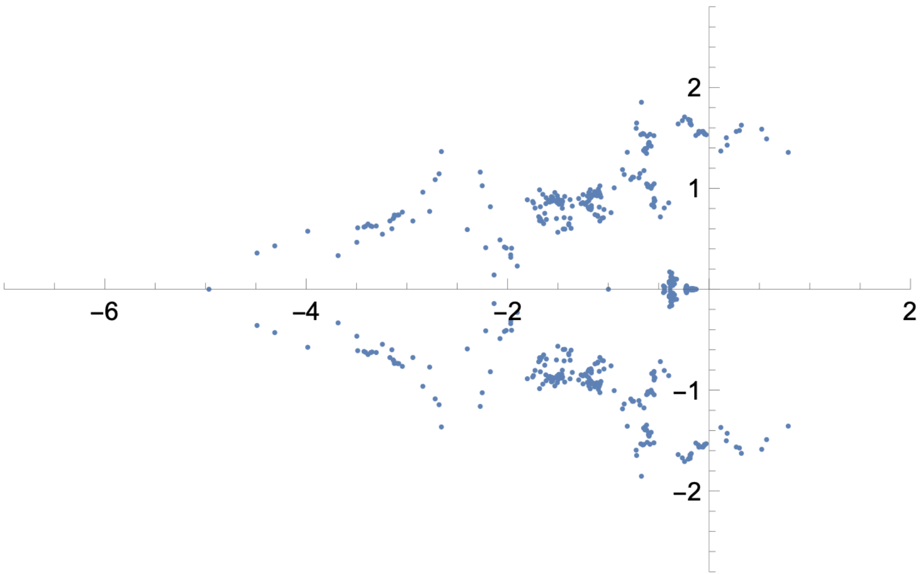


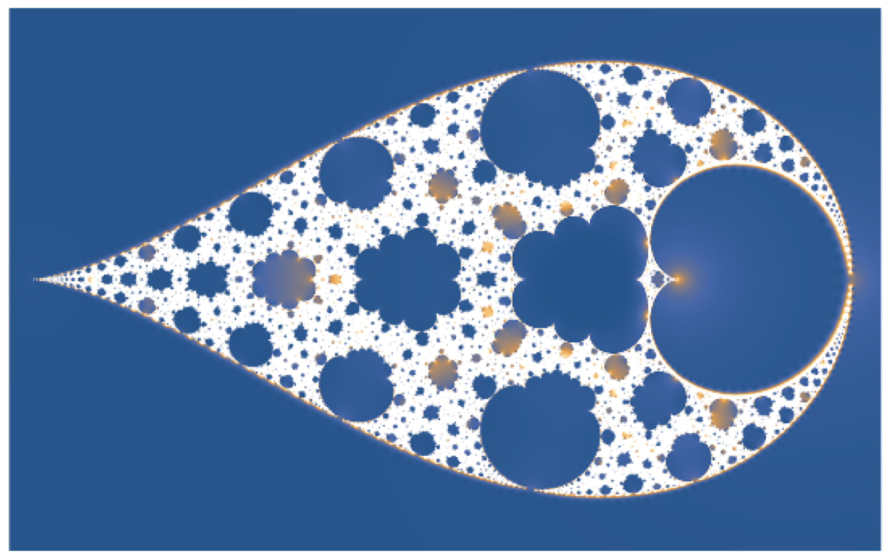

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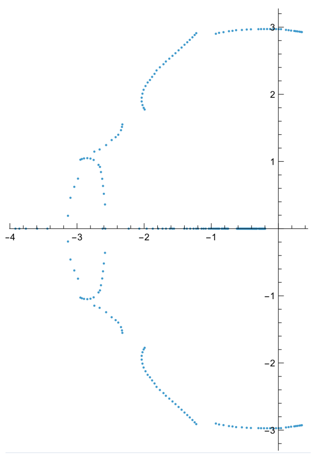

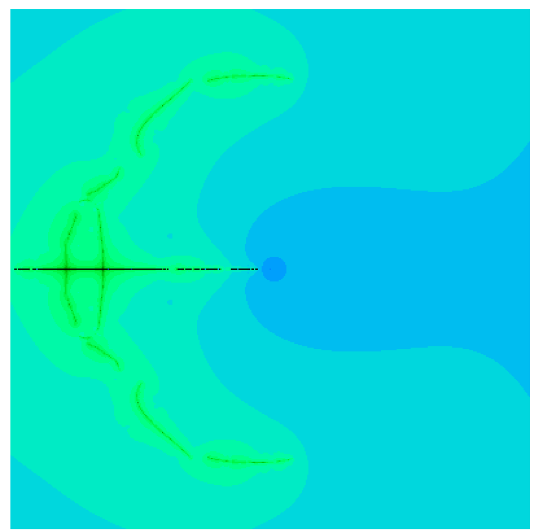

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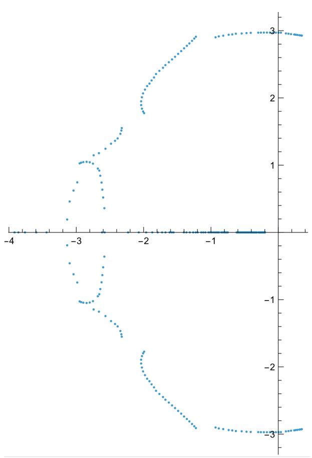


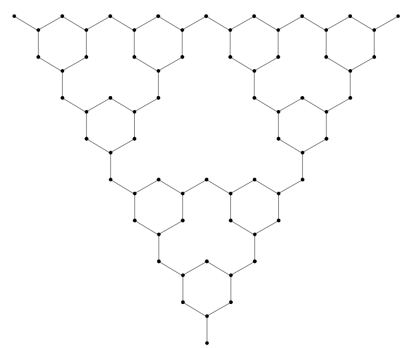

**THANK YOU for your attention!**