Pirogov-Sinai Beyond Lattices

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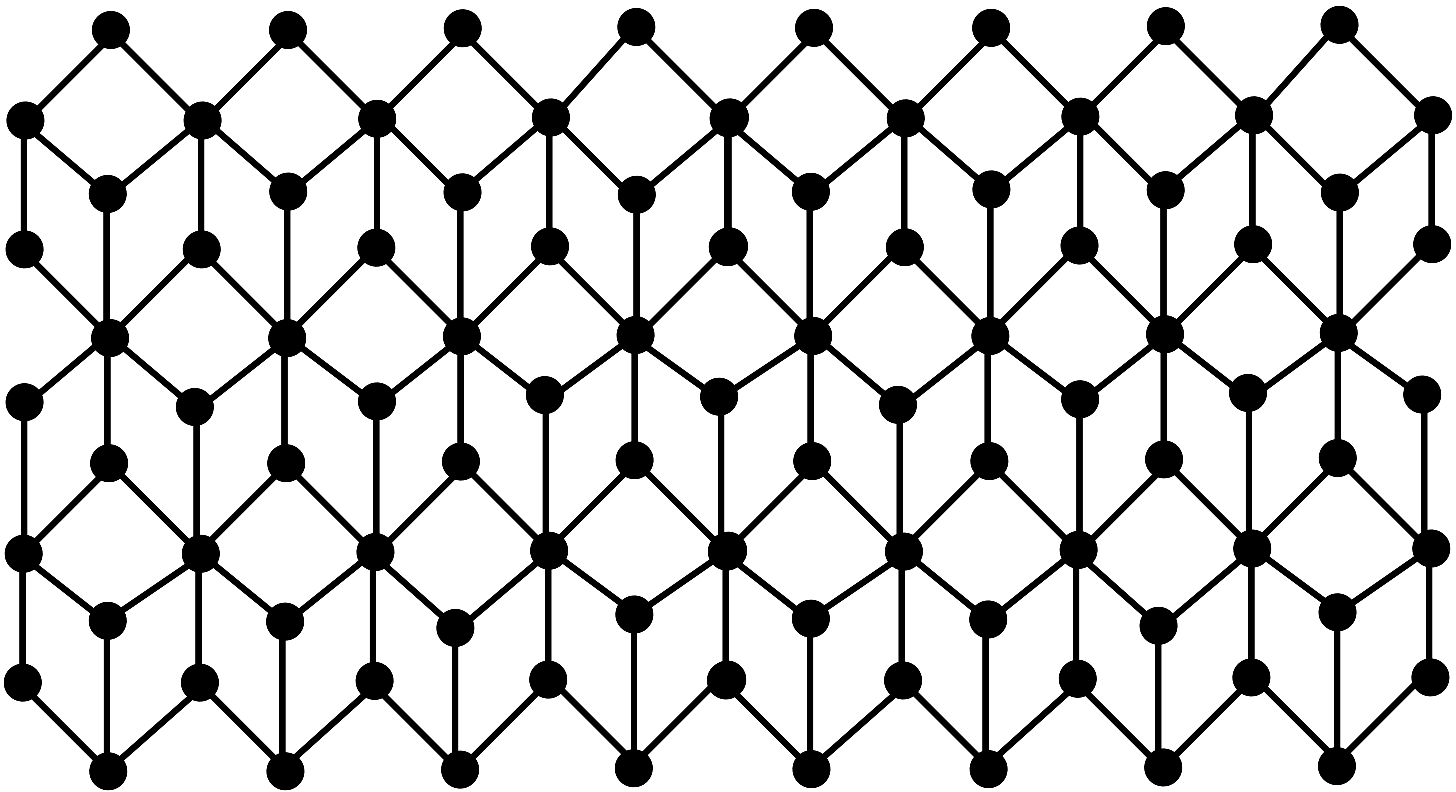

Workshop Theme: Describe $Z(G,\lambda) = \sum_{I \subset V} \lambda^{|I|} \mathbb{1}_{\{I \text{ independent}\}}$

Viewpoint: $Z(\cdot,\cdot)$ a map on $\tilde{\mathcal{G}} \subset \{\text{graphs}\}$, $\lambda \in D \subset \mathbb{C}$.

Depending on $\tilde{\mathcal{G}}$, D, properties of $Z(\cdot,\cdot)$?

Questions:
- Algorithm to (approx.) compute?
- Locations of zeros?
- "main contributions" / $\mathbb{P}_\lambda(I) \propto \lambda^{|I|}$?

Computational View: $\tilde{\mathcal{G}}$ huge, e.g., $\{\text{max. deg. } \Delta \text{ graphs}\} \equiv \mathcal{G}_\Delta$

λ fixed, e.g., $\lambda = 1$.

Physical View: $\tilde{\mathcal{G}}$ small, e.g., $\{\text{boxes in } \mathbb{Z}^d\} \equiv \mathcal{G}_{\mathbb{Z}^d}$

λ varies, e.g., $\lambda \geq 0$, $\lambda \gg 1$.

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"Facts":

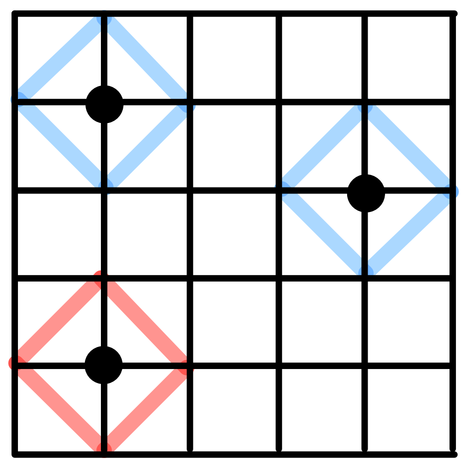
- $\mathcal{G}_{\Delta,\text{bip}} = \{\text{bipartite max deg. } \Delta\}$ very interesting; not understood.
- Physical understanding $\rightsquigarrow$ computational understanding

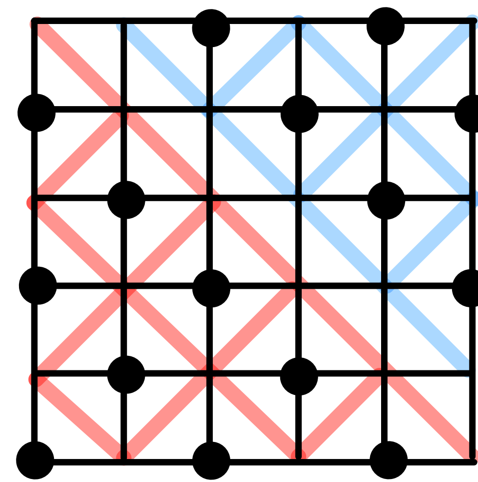
¿ HOW TO RELAX / GO BEYOND SMALL FOR $\lambda \gg 1$?

PHYSICAL UNDERSTANDING: G infinite, bipartite. $G_n \uparrow G$ finite.

Question: Describe the set $\text{Gibbs}_\lambda = \{\text{weak limits of } \mathbb{P}_{G_n,\lambda}\}$.

$\mathbb{Z}^d$, $d \geq 2$:





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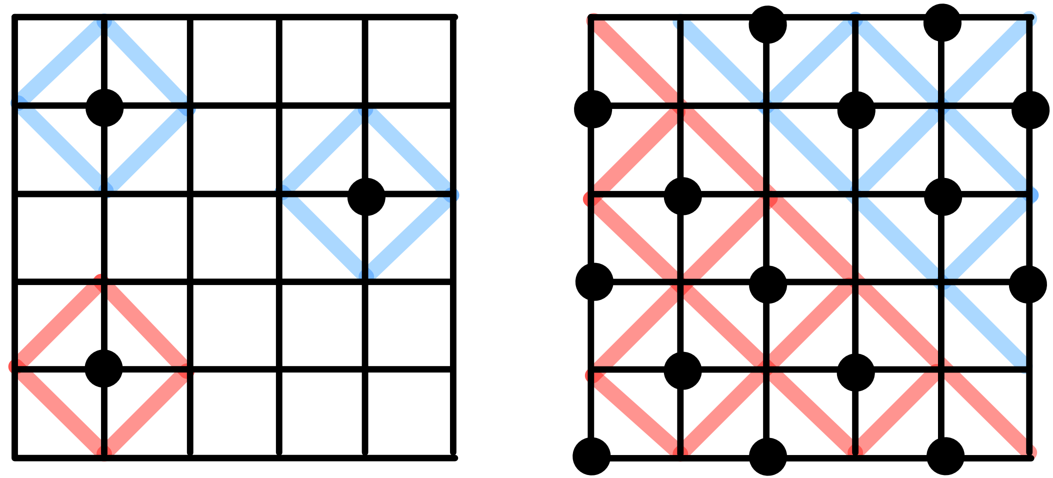
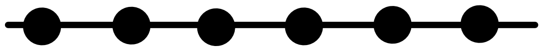


Theorem: (Dobroshin) $\underbrace{|\mathrm{Gibbs}_\lambda| = 1 \text{ if } \lambda \ll 1}_{\text{cluster expansion}}$; $\underbrace{|\mathrm{Gibbs}_\lambda| > 1 \text{ if } \lambda \gg 1}_{\text{Peierls/PS-Theory}}$.

PHYSICAL UNDERSTANDING: Important counterexamples ($\lambda \gg 1$)

1) 

- 1-dim examples have ! limit.

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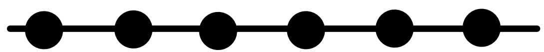
1) 

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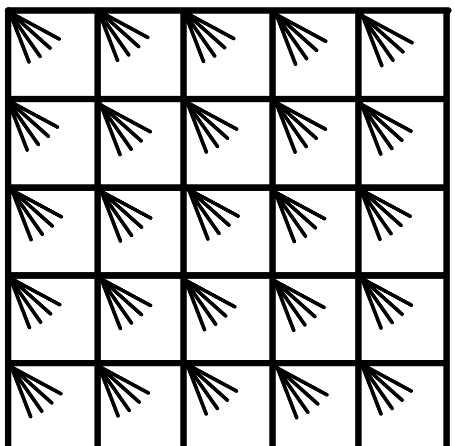
2) 

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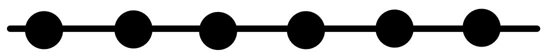
1) 

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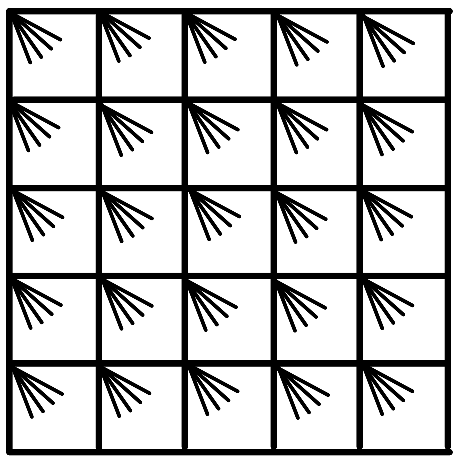
2) 

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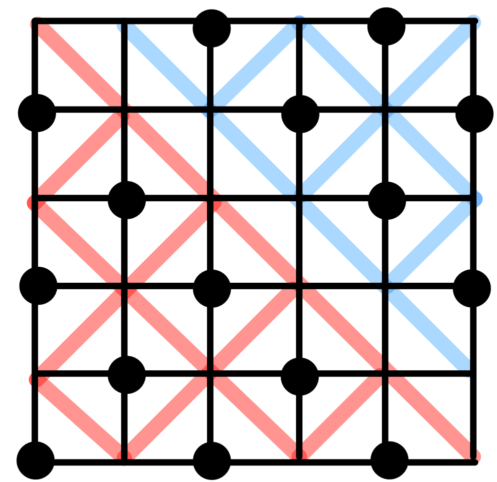
3) 

• open ($d \geq 3$) to show $\lambda_{\text{red}} \neq \lambda_{\text{blue}}$ implies $|\mathrm{Gibbs}_{\lambda_{\text{red}}, \lambda_{\text{blue}}}| = 1$.

PHYSICAL UNDERSTANDING:

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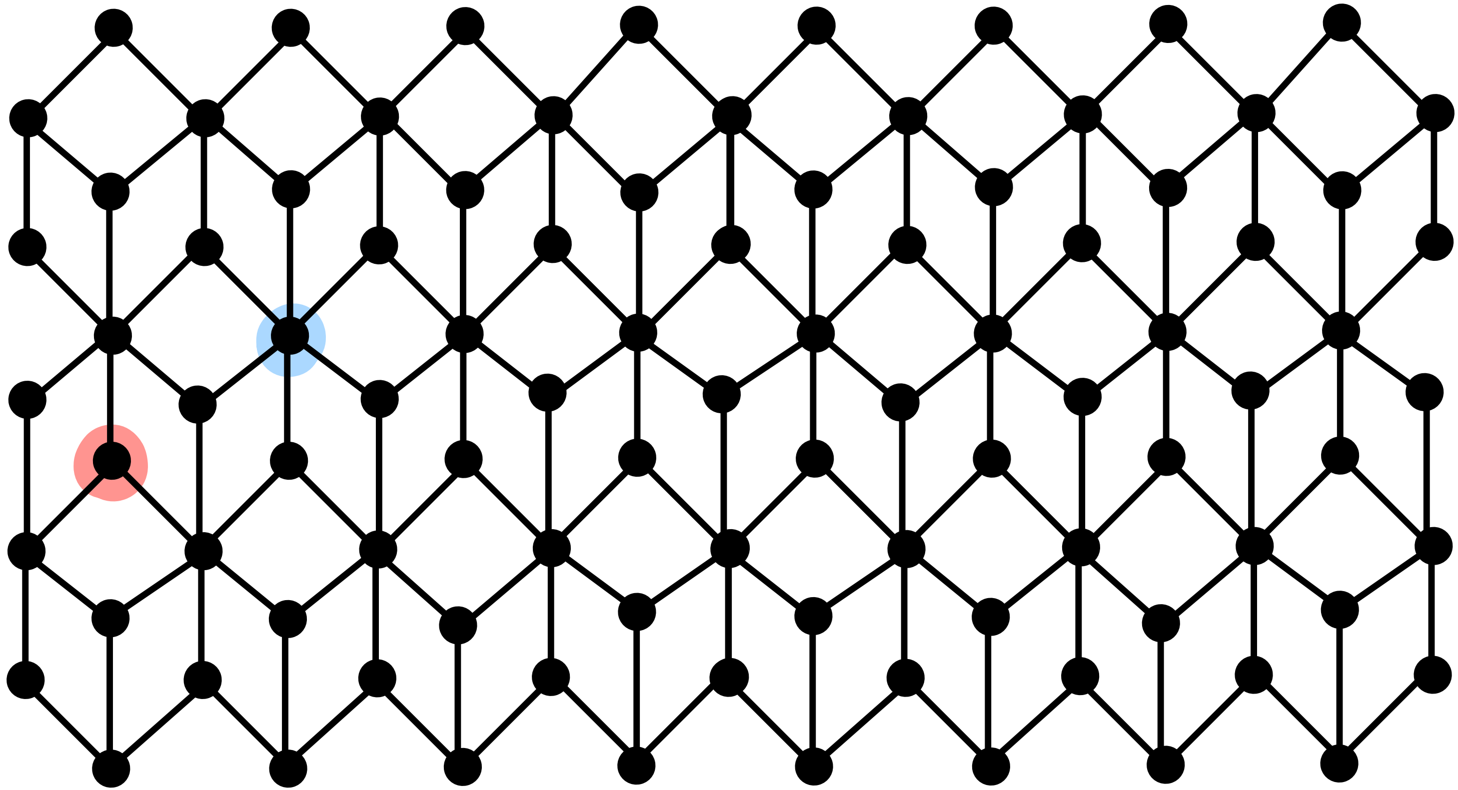
• If G is vertex-transitive within a class $\exists$ $\lambda_{o,c} = \lambda_{o,c}(\lambda_e)$ s.t. $|\mathrm{Gibbs}_{\lambda_{o,c}, \lambda_e}| > 1$ if $\lambda_e \gg 1$

(and exactly one if $\lambda_o \neq \lambda_{o,c}$ in interval around $\lambda_{o,c}$)

Ex: Dice lattice

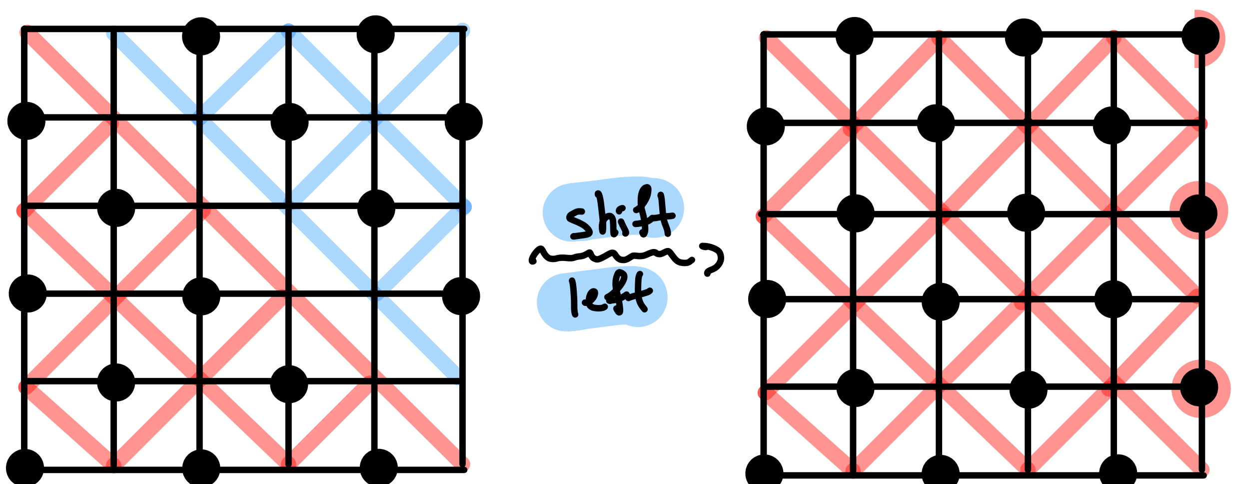


Coexistence line: $\log \lambda_6 = 2 \log \lambda_3 + 6\lambda_3^{-1} + o(\lambda_3^{-1})$

REMARKS: • FAR from satisfactory – symmetry hypotheses very strong.
– bounded cycle basis very strong.

• Algorithmic results follow (for nice subgraphs)

IDEAS: ① Dobrushin understood spatial symmetry:



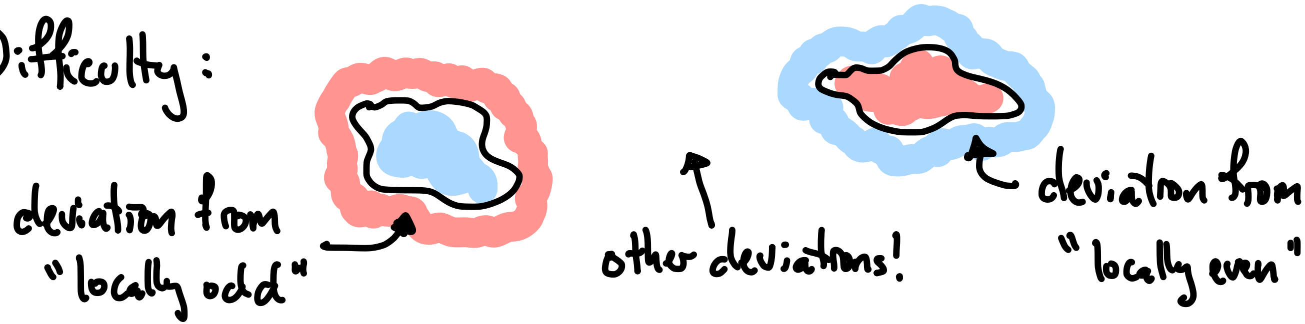
• gain in prob. (Peierls) enough to show coexist.

• not enough if not transitive

(2) $\lambda \ll 1$ is $\approx$ Poisson (this is what the cluster expansion says)

$\lambda \gg 1$ is $\overset{?}{\approx}$ Poisson for <u>deviations</u> from "locally even/odd"

Difficulty:



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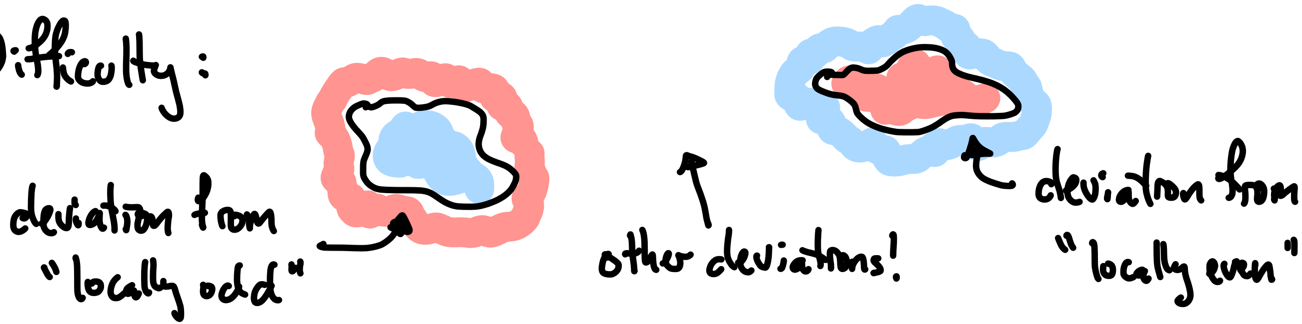

③ <u>Sol^{n}</u> (PSZ)

Pirogov-Sinai-Zahradnik

a) **Euclidean facts** ("inside") can hide this, at price of complicated weights.

b) **Euclidean facts** ("isoperimetry") allow for inductive control of complicated weights.

Ⓐ (Inspired by Timár, Georgakopoulos–Panagiotis)

Replace all appeals to Euclidean facts by combinatorial ideas:

one-endedness + choice of cycle basis.

Main result: bijection b/w IS and contour configurations $\Gamma = \{\gamma\}$

basis-connected sets γ of edges s.t. components of $G \setminus \gamma$ have vertices incident to γ of only one parity.

Ⓑ Symmetry conditions replace "spatial shift" – and isoperimetry conditions follow for free (deep structure theory; Gromov...)

Going further:

- Weaken symmetry hypotheses?
- Uniform control on what $\lambda \gg 1$ means, e.g., for Cayley graphs?
- Growing sequences of finite graphs?
- Relax boundedness of cycle basis?

THANKS!