Sampling from the random-cluster model on random regular graphs

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Joint with:

Andreas Galanis, Paulina Smolarova

(their lovely slides!)

Combinatorial, Algorithmic and Probabilistic aspects of Partition Functions

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This talk: connections between

- structural properties

  What does a typical sample from the distribution look like?

- dynamical properties

  Can we sample efficiently using Markov chains?

Random Cluster Model

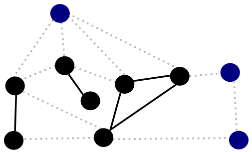
graph $G = (V, E)$, $q \geq 1$, $\beta > 0$ reals

Configuration set Ω_G: edge subsets $F \subseteq E$

Weights: $w_G(F) = q^{c(F)}(\mathrm{e}^{\beta} - 1)^{|F|}$ where $c(F) =$# components in (V, F)

Gibbs distribution: $\pi_G(F) = w_G(F)/Z_G$

Partition function: $Z_G = \sum_{F \subseteq E} w_G(F)$



RC configuration with $c(F) = 6$, $|F| = 5$

For integer $q \geq 2$: Random Cluster $\longleftrightarrow$ Potts
Potts-model configurations: $\sigma : V \to \{1, \ldots, q\}$.

Weight: factor of e^{β} for each monochromatic edge.

Sampling using Glauber dynamics

Markov chain $(X_t)_{t\geq 0}$ on edge subsets

Roughly: update whether $e \in X_t$ conditioned on the status of $E\backslash\{e\}$.

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Start from arbitrary X_0. To obtain X_{t+1} from X_t:

1. Pick an edge $e \in E$ u.a.r.
2. With probability $\dfrac{w_G(X_t \cup \{e\})}{w_G(X_t \cup \{e\}) + w_G(X_t \setminus \{e\})}$, set $X_{t+1} = X_t \cup \{e\}$

   Otherwise, $X_{t+1} = X_t \setminus \{e\}$

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Standard fact: X_t converges to π_G

T_{mix}=# steps to get within TV distance $\leq 1/4$ from π_G

Question: fast vs slow mixing?

Computational Complexity Results

Question: On input $G = (V, E)$, can we sample from π_G in $poly(|V|)$ time?

[Jerrum-Sinclair '93, Guo-Jerrum '17]:

Poly-time algorithm when $q = 2$ and $\beta > 0$

(JS: even-subgraphs formulation; GJ: edge-flip dynamics for random cluster (and SW))

[Goldberg-Jerrum '10]:

Hard when $q > 2$ and $\beta > 0$

#BIS-hard

[Galanis-Stefankovic-Vigoda-Yang '14]:

For int $q \geq 3$, hard on Δ-regular graphs when $\beta > \beta_c := \ln \frac{q-1}{(q-2)^{1-2/\Delta} - 1}$

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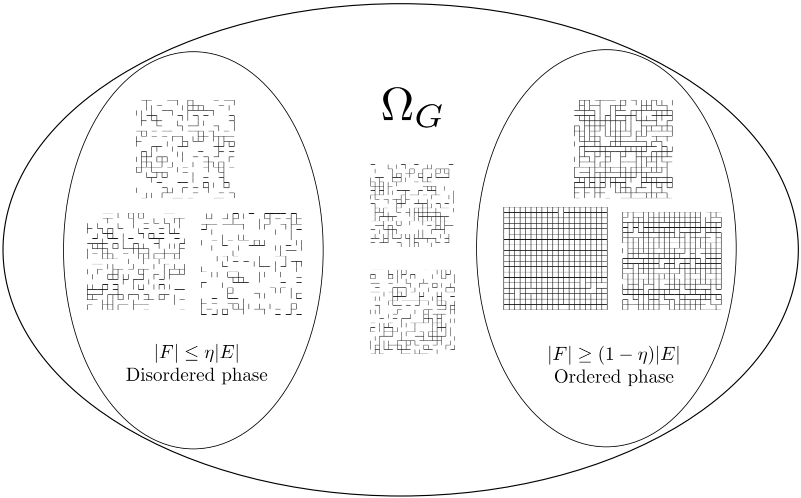
Can we understand behaviour of Glauber on the random regular graph?

- Same phenomena believed to be relevant even for worst-case graphs
- Well studied in the mean-field case (complete graph)
- [Bollobás-Grimmett-Janson '96], [Gore-Jerrum '96], [Cooper-Dyer-Frieze-Rue '06], [Long-Nachmias-Ning-Peres '14], [Cuff-Ding-Louidor-Lubetzky-Peres-Sly '12], [Galanis-Stefankovic-Vigoda '17], [Blanca-Sinclair '17], [Gheissari-Lubetzky-Peres '18]

Phases for RC on random regular graphs

Disordered vs Ordered Configurations: $|F| \le \eta|E|$ vs $|F| \ge (1-\eta)|E|$



A typical $F \sim \pi_G$ is w.h.p. either *ordered* and *disordered*

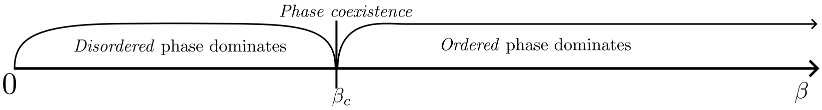
Phase transition on the random regular graph

[Helmuth-Jenssen-Perkins '20] (q large)

Let $\eta = \eta(\Delta) > 0$ be a small constant. Define

- Disordered phase Ω^{dis}: $\{F \subseteq E \text{ with } |F| \leq \eta|E|\}$
- Ordered phase Ω^{ord}: $\{F \subseteq E \text{ with } |F| \geq (1-\eta)|E|\}$

Let $\pi^{\text{dis}} = \pi(\cdot \mid \Omega^{\text{dis}})$, $\pi^{\text{ord}} = \pi(\cdot \mid \Omega^{\text{ord}})$



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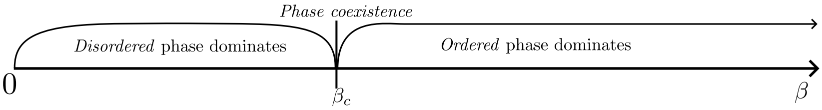

Phase transition at $\beta_c = (1 + o_q(1))\frac{2 \log q}{\Delta}$

- if $\beta < \beta_c$, $|\pi - \pi^{\text{dis}}| = e^{-\Omega(n)}$
- if $\beta > \beta_c$, $|\pi - \pi^{\text{ord}}| = e^{-\Omega(n)}$
- for $\beta = \beta_c$: $\pi(\Omega^{\text{dis}}), \pi(\Omega^{\text{ord}}) = \Omega(1)$, $\pi(\Omega^{\text{dis}} \cup \Omega^{\text{ord}}) = 1 - e^{-\Omega(n)}$.

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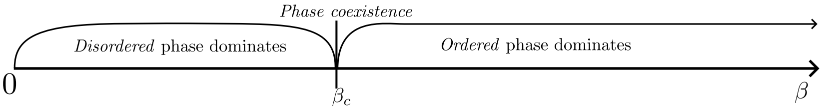

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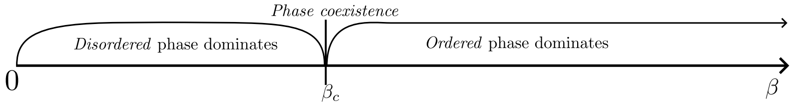
[Galanis-Stefankovic-Vigoda-Yang '14], [Bencs-Borbényi-Csikvári '22]:

$\beta_c = \ln \frac{q-1}{(q-2)^{1-2/\Delta} - 1}$ for all $q > 2$

Glauber on the random regular graph

What about Glauber dynamics?



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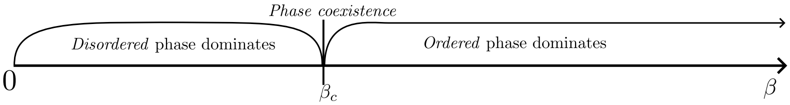
What about Glauber dynamics?

Two other thresholds $\beta_u < \beta_c < \beta_h$ that are relevant

- connected to uniqueness threshold on infinite Δ-regular tree
- [Haggström '96]: showed non-uniqueness (on tree for $q > 2$) when $\beta \in [\beta_u, \beta_h]$



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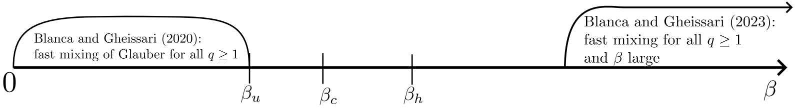

◦ [Blanca-Gheissari '20]: for $\beta < \beta_u$, $T_{\text{mix}} = \Theta(n \log n)$ for $q \geq 1$

◦ [Blanca-Gheissari '23]: for large β, $T_{\text{mix}} = \Theta(n \log n)$ for $q \geq 1$

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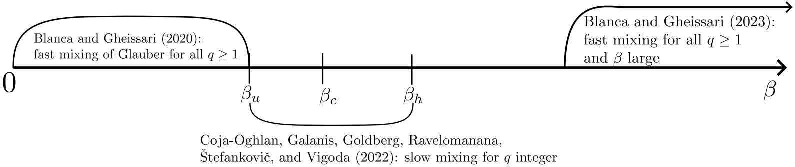

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○ [Coja Oghlan-Galanis-Goldberg-Ravelomanana-Stefankovic-Vigoda '21]

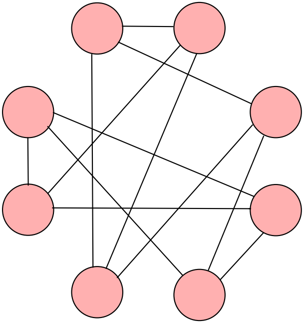
For integer $q \geq 3$, metastability for $\beta \in (\beta_u, \beta_h)$

Exponential time for the dynamics to move from Ω^{ord} to Ω^{dis} (and vice versa)

Ordered/Disordered on Random Regular graphs

For $a \in (0,1)$: $\Omega_G(a) = \{F \subseteq E : |F| = a|E|\}$

random Δ-regular graph



$$Z_G(a) = \sum_{F \in \Omega_G(a)} w_G(F)$$

$$Z_G = \sum_a Z_G(a)$$

Which a achieve $\max_a Z_G(a)$?

Plots of $\Psi(a) := \frac{1}{n} \log Z_G(a)$ for integer q where $Z_G = \sum_a e^{n\Psi(a)}$

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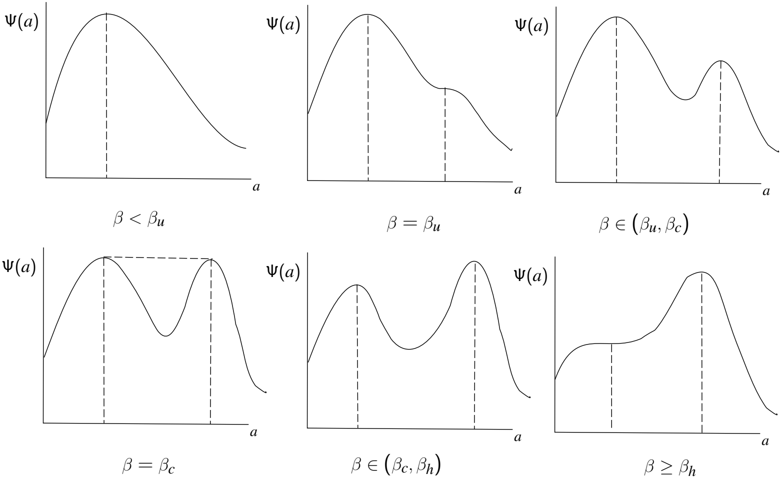

- $\beta < \beta_u$: disordered is the only local max
- $\beta_u < \beta < \beta_c$ both local max, disordered is the global max
- $\beta_c < \beta < \beta_h$ both local max, ordered is the global max
- $\beta > \beta_h$ ordered is the only local max

[Helmuth-Jenssen-Perkins '20:]

Poly-time algorithm when q is large for all β

Based on cluster-expansion methods (expanding power series of partition fn)

Idea: Approx Z_G^{dis} for $\beta < \beta_1$, Z_G^{ord} for $\beta > \beta_0$

Crucially: $\beta_0 < \beta_c < \beta_1$

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Question: Can we obtain a fast algorithm using Glauber?

◦ Perhaps we can avoid bottlenecks from well-chosen starting configurations

- For $\beta < \beta_c$, start from all-out ($X_0 = \emptyset$)
- For $\beta > \beta_c$, start from all-in ($X_0 = E$)
- For $\beta = \beta_c$, start from appropriate mixture of $\emptyset$ and E.

Intuition: Glauber should mix well within Ω^{ord} and Ω^{dis}

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[Gheissari-Sinclair '21]:

Obtained analogous starting-state result for Ising ($q = 2$) but for β large

Our Result

Let $\Delta \geq 5$. $\exists C = C(\Delta)$ s.t. for all q large enough and any $\beta > 0$, w.h.p. over random Δ-regular graph:

1. For $\beta < \beta_c$, T_{mix} of Glauber starting from all-out is $O(n \log n)$.
2. For $\beta > \beta_c$, T_{mix} of Glauber starting from all-in is $O(n^C)$.

   For integer q, the mixing time is $O(n \log n)$.

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Notes:

◦ For $\beta = \beta_c$, analogous T_{mix} bound from all-in/all-out mixture

◦ Proof builds on cluster expansion results of [Jenssen-Helmuth-Perkins '20]

Proof sketch ($\beta > \beta_c$)

X_t: Glauber from all-in

$\hat{X}_t$: Glauber from stationarity, but restricted to ordered phase

Goal: For any edge e, $|\Pr(e \in X_t) - \Pr(e \in \hat{X}_t))| \leq 1/(4|E|)$

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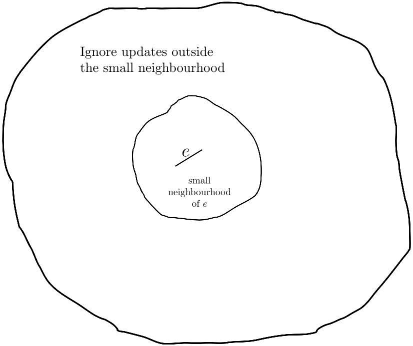
Proof technique: Use local behaviour to extrapolate something global

◦ Can we show edges far away have little influence on e?

◦ Key: compare with a chain restricted to a *small neighbourhood* around e

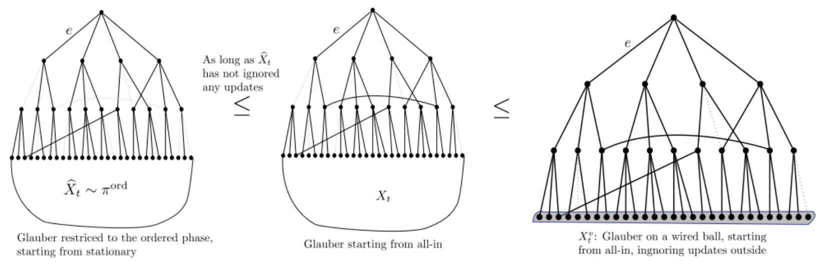


Proof sketch: edge marginals

Pick $v \in V$ incident to e, $r = \Theta(\log n)$

(X_t^v): restricted chain starting from all-in, ignoring updates outside $B_r(v)$

(X_t^v) converges to $\pi_{B_r^+(v)}$ (RC on the wired ball)



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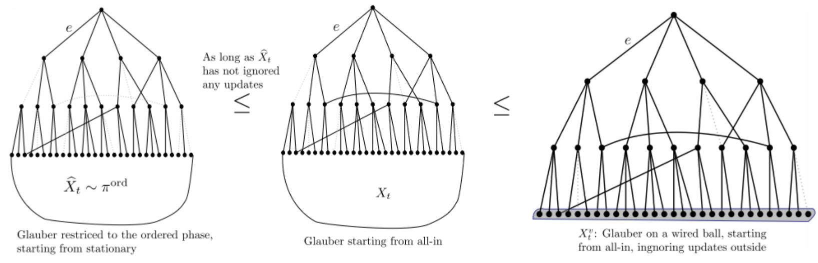

$$|\Pr(e \in X_t^v) - \Pr(e \in \hat{X}_t)| \leq \quad \text{(triangle inequality)}$$

$$\leq |\Pr(e \in X_t^v) - \pi_{B_r^+(v)}(e \in \cdot)| + |\pi_{B_r^+(v)}(e \in \cdot) - \Pr(e \in \hat{X}_t)|$$

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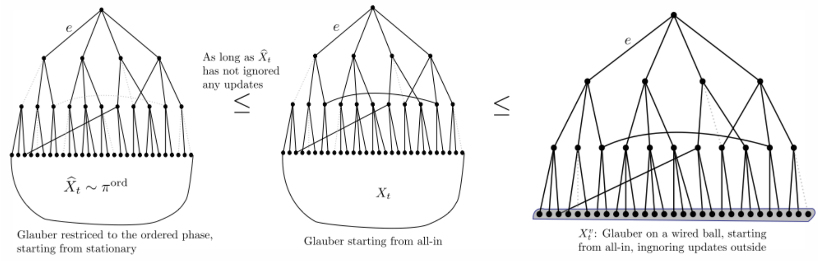

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For t poly, prob of ignoring updates is small, LHS conditioned on this is at most
$|\,\mathrm{Pr}(e \in X_t^v) - \pi_{B_r^+(v)}(e \in \cdot)| + |\pi_{B_r^+(v)}(e \in \cdot) - \pi^{\mathrm{ord}}(e \in \cdot)| + \mathrm{e}^{-\Omega(n)}$