Occupancy fractions: graph theory and algorithms

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Workshop on the Combinatorial, Algorithmic and Probabilistic aspects of Partition Functions

1. What is the smallest independence number of an n-vertex graph of average degree d?

A: Turán 1941: $\frac{n}{d+1}$ achieved by a disjoint union of K_{d+1}s

2. Which d-regular n-vertex graph has the most independent sets?

A: Kahn 2001, Zhao 2010: a disjoint union of $K_{d,d}$s

3. Which graph with given degree sequence has the fewest independent sets?

A: SSSZ 2019: a disjoint union of complete graphs

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1. In expectation, how big is the independent set returned by the random greedy algorithm?

A: Folklore (Alon–Spencer 1992?): at least $\sum_v \frac{1}{d_v+1}$

2. In which graphs can you efficiently approximate the number of independent sets of density 1/8?

A: D., Perkins 2021: graphs of maximum degree 5

3. When can you efficiently approximate the antiferromagnetic Ising partition function at a given magnetization?

A: Not known, partial knowledge for cubic graphs (D., Leblanc 2025)

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Let $E_G(1)$ be the expected density of a uniform random independent set in G

1. Which graph of maximum degree Δ minimizes $E_G(1)$?

A: Cutler, Radcliffe 2014: $K_{\Delta+1}$

2. Which $(2d-1)$-regular line graph maximizes $E_G(1)$?

A: DJPR 2017a: $L(K_{d,d})$

3. Over triangle-free graphs of maximum degree Δ what is the infimum of $E_G(1)$?

A: Shearer[1], DJPR 2017b: at least $(1-o(1))\frac{\log \Delta}{\Delta}$

[1] Unpublished, 1992 SIAM talk

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- Let $Z_G(\lambda)$ be the independence polynomial:

$$Z_G(\lambda) = \sum_{I \subset V(G) \text{ indep.}} \lambda^{|I|}$$

- Let $F_G(\lambda)$ be the free energy density:

$$F_G(\lambda) = \frac{1}{|V(G)|} \log Z_G(\lambda)$$

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- Let $\mu_{G,\lambda}(I)$ be the hard-core model on G

$$\mu_{G,\lambda}(I) = \frac{\lambda^{|I|}}{Z_G(\lambda)}$$

- Let $E_G(\lambda)$ be the occupancy fraction of G:

$$E_G(\lambda) = \frac{1}{|V(G)|}\mathbb{E}_{\mu_{G,\lambda}}|I| = \lambda\frac{\partial}{\partial\lambda}F_G(\lambda) = \frac{1}{|V(G)|}\frac{\lambda Z'_G(\lambda)}{Z_G(\lambda)}$$

In what ways can partition functions be bigger than others?

$P(x) = 1 + a_1 x + \ldots + a_k x^k$; $Q(x) = 1 + b_1 x + \ldots + b_k x^k$; $a_i, b_j \geqslant 0$.

Definition

1. Say $P \geqslant_{\text{MAX}} Q$ if $a_k \geqslant b_k$
2. Say $P \geqslant_{\text{COUNT}} Q$ if $\sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} b_i$ — $F_P(1) \geqslant F_Q(1)$
3. Say $P \geqslant_{\text{PART}} Q$ if $P(x) \geqslant Q(x)$ for all $x \geqslant 0$ — $F_P(x) \geqslant F_Q(x)$
4. Say $P \geqslant_{\text{OCC}} Q$ if $\frac{xP'(x)}{P(x)} \geqslant \frac{xQ'(x)}{Q(x)}$ for all $x \geqslant 0$ — $E_P(x) \geqslant E_Q(x)$
5. Say $P \geqslant_{\text{COEF}} Q$ if $a_i \geqslant b_i$ for all $1 \leqslant i \leqslant k$
6. Say $P \geqslant_{\text{FV}} Q$ if $b_i a_{i+1} \geqslant a_i b_{i+1}$ for all $0 \leqslant i \leqslant k-1$

Theorem (DJPR 2018)

$$
\begin{array}{ccccc}
 & & \text{FV} & & \\
 & \Swarrow & & \Searrow & \\
\text{COEF} & & \begin{array}{c}\not\Rightarrow \\ \not\Leftarrow\end{array} & & \text{OCC} \\
 & \Searrow & & \Swarrow & \\
 & & \text{PART} & & \\
 & \Swarrow & & \Searrow & \\
\text{COUNT} & & \begin{array}{c}\not\Rightarrow \\ \not\Leftarrow\end{array} & & \text{MAX}
\end{array}
$$

Lens through which we can see combinatorial results

- Over d-regular graphs K_{d+1}, minimizes $Z_G(\lambda)$ in the FV sense [Cutler, Radcliffe 2014]
- For a given degree sequence, a disjoint union of complete bipartite graphs maximizes $Z_{L(G)}(\lambda)$ in the MAX sense [Bregman 1973]
- For a given degree sequence, a disjoint union of complete bipartite graphs maximizes $Z_G(\lambda)$ in the PART sense [SSSZ 2019]
  Fact: strengthening to OCC is *false*
- For a given degree sequence, a disjoint union of complete graphs minimizes $Z_G(\lambda)$ in the PART sense [SSSZ 2019]
  Conj. [D., Kang]: strengthening to OCC is *true*

Motivation

Theorem (DJPR 2017)

For a triangle-free graph G of maximum degree Δ,

$$E_G(\underbrace{1/\log\Delta}_{\lambda}) \geqslant (1-o(1))\frac{\log\Delta}{\Delta}$$

- This is tight as stated by the random Δ-regular graph
- But what about $\lambda \to \infty$? Then we seem to be a factor 2 off
- We know $E_G(\lambda)$ is increasing in λ, but how fast?
- Perhaps we should study the derivative...

Variance fraction

- Let $V_G(\lambda)$ be the variance fraction of G:

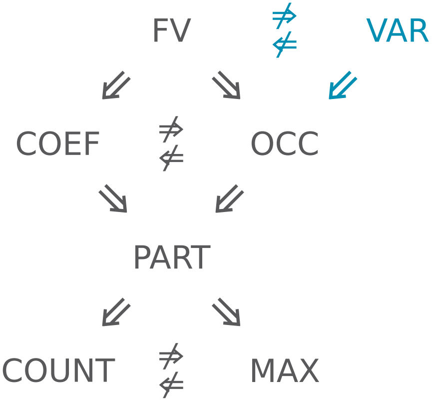
$$V_G(\lambda) = \frac{1}{|V(G)|}\mathbb{V}_{\mu_{G,\lambda}}|I|$$
$$= \lambda \frac{\partial}{\partial \lambda} E_G(\lambda) = \frac{1}{|V(G)|}\left(\frac{\lambda^2 Z_G''(\lambda) + \lambda Z_G'(\lambda)}{Z_G(x)} - \frac{\lambda^2 Z_G'(\lambda)^2}{Z_G(\lambda)^2}\right)$$

- For polynomials P, Q as before, we say that $P \geqslant_{\mathrm{VAR}} Q$ if

$$\frac{x^2 P''(x) + xP'(x)}{P(x)} - \frac{x^2 P'(x)^2}{P(x)^2} \geqslant \frac{x^2 Q''(x) + xQ'(x)}{Q(x)} - \frac{x^2 Q'(x)^2}{Q(x)^2}$$

for all $x \geqslant 0$

General results



Perhaps this is a little surprising, FV seemed pretty strong. . .

New results for the hard-core model

Theorem (D., Sandhu, Tan 2025)

- Let G be an n-vertex graph. Then for any $0 < \lambda < 1/(2n-1)$ we have

$$\frac{\lambda}{(1+n\lambda)^2} = V_{K_n}(\lambda) \leqslant V_G(\lambda) \leqslant V_{\overline{K_n}}(\lambda) = \frac{\lambda}{(1+\lambda)^2},$$

  and the upper bound holds up to $\lambda \leqslant 1/n$.
  (not quite VAR due to upper bound on λ)

- It is true that

$$Z_{3K_2+K_3}(\lambda) \leqslant_{\mathrm{VAR}} Z_{3K_{1,2}}(\lambda)$$

  (first nontrivial case of deg-seq VAR lower bound)

New results for the hard-core model

Theorem (D., Sandhu, Tan 2025)

Let G be an n-vertex graph with maximum degree Δ. Then for any $0 < \lambda < 3/(\Delta+1)^2$ we have

$$\frac{1}{n}\sum_{u\in V(G)} \frac{\lambda}{1+(d_u+1)\lambda} = \frac{1}{n}\sum_{u\in V(G)} E_{K_{d_u+1}}(\lambda) \leqslant E_G(\lambda).$$

(progress on OCC strengthening of SSSZ deg-seq PART bound)

Note that the bound holds in the limit $\lambda \to \infty$ because it's equivalent to the Caro–Wei theorem (1978, 1981)

Claim: the proof techniques are interesting

- For VAR: formulate a higher-order version of occupancy method techniques which yielded many results for OCC
- Objective is now quadratic instead of linear
- We showed that it can work, but it's unclear how sharp these local methods are. Is variance well-explained locally?
- For deg-seq OCC: use an induction approach (cf. SSSZ) with extension of local occupancy (cf. DJPR, DJKP, DKPS, DK)
- Lots of things are tricky, Taylor expansions get out of hand...
- $\lambda = O(1/\Delta)$ seems like a serious barrier for our approach
- So far, methods are combined but not 'properly' interwoven so perhaps more can be done

Open problems

- Understand variance for large λ in any setting
- Extend the range of λ in our results
- Entertain an interest in more parameters:

Theorem (Campos–Samotij 2024)

For all graphs G and $\lambda > 0$, $E_G(\lambda) \leqslant F_G(\lambda)\frac{\lambda}{(1+\lambda)\log(1+\lambda)}$

Stronger than $\overline{K_n}$ maximizing in the OCC sense!

- Take the derivative $\lambda\frac{\partial}{\partial\lambda}$:

Conjecture (D., Sandhu, Tan 2025)

For all graphs G and $\lambda > 0$, $V_G(\lambda) \leqslant E_G(\lambda)\frac{1}{1+\lambda}$

Stronger than $\overline{K_n}$ maximizing in the VAR sense!

Thank you

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