Random cluster model on locally tree-like regular graphs

Free energy and typical landscape

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Ongoing work with Ferenc Bencs and Péter Csikvári.

Plan

1. Definitions and setup
2. Results
3. Idea of the proof

Random cluster model

Let $p \in (0,1)$ and $q > 0$ be fixed. The probability that $A \subseteq E(G)$ is chosen is

$$\mathbb{P}[A] \propto p^{|A|}(1-p)^{|E(G)\setminus A|}q^{k(A)},$$

where $k(A)$ denotes the number of connected components of (V, A).

(a) $\mathbb{P}[A_1] \propto p^4(1-p)^4q^2$ (b) $\mathbb{P}[A_2] \propto p^5(1-p)^3q$

Alternatively,

$$\mathbb{P}[A] \propto q^{k(A)}\Bigg(\underbrace{\frac{p}{1-p}}_{w}\Bigg)^{|A|}.$$

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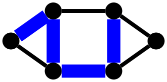

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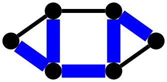

(b) $\mathbb{P}[A_2] \propto p^5(1-p)^3q$

Alternatively,

$$\mathbb{P}[A] \propto q^{k(A)}\underbrace{\left(\frac{p}{1-p}\right)}_{w}^{|A|}.$$

For a graph G we denote the partition function of the random cluster model as

$$Z_G(q,w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|}.$$

Graphs

Let $(G_n)_n$ be an *essentially* large girth sequence of d-regular graphs.

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Examples:

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Examples:

- G_n is a random d-regular graph on n vertices
- G_{n+1} is a random double lift of G_n where G_0 is any d-regular graph.

Main questions

Question

What is the free energy on an essentially large girth sequence of d-regular graphs:

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What is local weak limit of this the random cluster model on G_n? How does a typical configuration look like?

Previous results

- Dembo, Montanari / Montanari, Mosser, Sly:
  Ising model $(q = 2)$
- Dembo, Montanari, Sun:
  Potts model (positive integer q), except an interval (w_0, w_1)
- Dembo, Montanari, Sly and Sun:
  Potts model (positive integer q) and even d
- Helmuth, Jenssen and Perkins:
  random cluster model with large q and assuming some expansion property of $(G_n)_n$
- Basak, Dembo and Sly:
  local structure of the Potts model (positive integer q) for all d

Answers

Theorem (BBCs '22)

If $(G_n)_n$ is an essentially large girth sequence of d-regular graphs, then the limit

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}(q,w) = \ln \Phi_{d,q,w}$$

exists for $q \geq 2$ and $w \geq 0$. The quantity $\Phi_{d,q,w}$ can be computed as follows. Let

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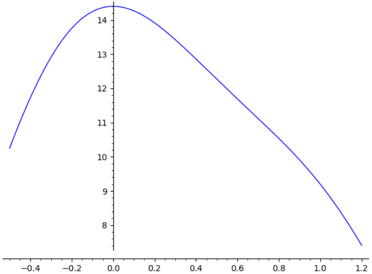
$$\Phi_{d,q,w}(t) := \left(\sqrt{1+\frac{w}{q}}\cos(t) + \sqrt{\frac{(q-1)w}{q}}\sin(t)\right)^d + (q-1)\left(\sqrt{1+\frac{w}{q}}\cos(t) - \sqrt{\frac{w}{q(q-1)}}\sin(t)\right)^d,$$

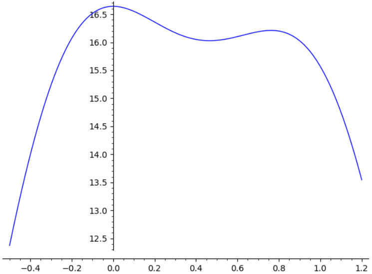
then

$$\Phi_{d,q,w} := \max_{t\in[-\pi,\pi]} \Phi_{d,q,w}(t).$$

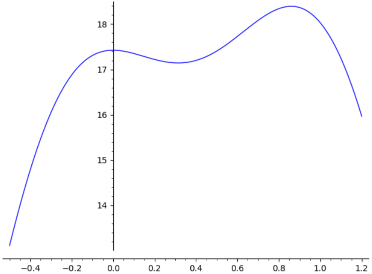
Examples



$q = 4, w = 2, d = 4$



$q = 4, w = 2.9, d = 4$



$q = 4, w = 3.2, d = 4$

Phase transition

Theorem (BBCs '22)

Let $q > 2$ *and*

$$w_c := \frac{q-2}{(q-1)^{1-2/d} - 1} - 1.$$

If $0 \le w \le w_c$, *then* $\Phi_{d,q,w} = q\left(1 + \frac{w}{q}\right)^{d/2}$.

If $w > w_c$, *then* $\Phi_{d,q,w} > q\left(1 + \frac{w}{q}\right)^{d/2}$.

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The model undergoes a first-order phase transition.

Local structure

Theorem (BBCs '25+)

Let $q > 2$.

- *If* $w < w_c$, *then the "typical" local structure of random cluster model on* G_n *"looks like" the free boundary condition Gibbs measure on* T_d.
- *If* $w > w_c$, *then the "typical" local structure of random cluster model on* G_n *"looks like" the wired boundary condition Gibbs measure on* T_d.

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Definition

For a configuration σ on G_n let $S_r\sigma$ be the statistics of the r-balls.

Theorem

For all $\varepsilon > 0$ and r there exists $c > 0$ such that

$$\mathbb{P}_{\sigma_n}\left[d_{TV}\left(S_r\sigma_n, \mu|_{B(o,r)}\right) \geq \varepsilon\right] < e^{-cv(G_n)}.$$

Idea

- Relate the random cluster model to the Ising model.

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- Ising model is well-understood.

Ising model

Let $\beta > 0, h \in \mathbb{R}$. For a $\sigma : V \to \{+, -\}$

$$\mathbb{P}[\sigma] = \frac{1}{Z_G(\beta, h)} e^{\beta \sum_{(u,v)\in E} \sigma_u \sigma_v + h \sum_{v \in V} \sigma_u},$$

where

$$Z_G(\beta, h) = \sum_{\sigma : V \to \{+,-\}} e^{\beta \sum_{(u,v)\in E} \sigma_u \sigma_v + h \sum_{v \in V} \sigma_u}.$$

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Ising as a spin model

$$N_{\text{Ising}} = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \text{ and } \nu_{\text{Ising}} = \begin{pmatrix} e^{h} \\ e^{-h} \end{pmatrix}.$$

The approximating spin model

2-spin approximation

Let

$$N = \begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix} \text{ and } \nu = \begin{pmatrix} 1 \\ q-1 \end{pmatrix}.$$

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In the d-regular case

$$\begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix}, \begin{pmatrix} 1 \\ q-1 \end{pmatrix} \sim \begin{pmatrix} 1+w & a \\ a & \left(1+\frac{w}{q-1}\right) a^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{q-1}{a^d} \end{pmatrix}$$

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Coloring the random cluster model

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Weight

$$w_{cRC}(A,\sigma) = \prod_{C \in \mathcal{C}_{cyclic}(A)} w^{e(C)} \cdot q \prod_{C \in \mathcal{C}_{tree}(A)} w^{e(C)} \cdot 1 \prod_{C \in \mathcal{C}_{tree}(A)} w^{e(C)} \cdot (q-1).$$

Remark

$$w_{RC}(A) = \sum_{\sigma} w_{cRC}(A,\sigma).$$

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Percolating the 2-spin model

1. Sample from the 2-spin model; percolate:

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Percolating the 2-spin model

1. Sample from the 2-spin model; percolate:
2. blue-red edge: remove;
3. red-red edge: keep with probability $\frac{w}{1+w}$,
4. blue-blue edge: keep with probability $\frac{\frac{w}{q-1}}{1+\frac{w}{q-1}} = \frac{w}{w+q-1}$ (active edges).

Weight

$$w_{pIs}(A,\sigma) = \prod_{C \in \mathcal{C}(A)} w^{e(C)} \cdot 1^{v(C)} \prod_{C \in \mathcal{C}(A)} \left(\frac{w}{q-1}\right)^{e(C)} \cdot (q-1)^{v(C)}$$

Modified Edwards–Sokal coupling

Weights

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cyclic	+	$w^{e(C)}$	$w^{e(C)}$**q**

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cyclic	-	$w^{e(C)}(q-1)^{v(C)-e(C)}$	0

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The number of cyclic components is sublinear $\left(\leq \frac{v(G_n)}{g}\right)$.

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Subexponentially close wieghts

$$w_{pIs}(A,\sigma) \leq w_{cRC}(A,\sigma) \leq w_{pIs}(A,\sigma) \cdot q^{\frac{v(G_n)}{g}}$$

Corollary

Theorem (BBCs '22)

$$\frac{1}{v(G_n)} \ln Z_{Is}(q,w) - \frac{1}{v(G_n)} \ln Z_{RC}(q,w) \to 0.$$

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$$\frac{1}{v(G_n)} \ln Z_{Is}(q,w) - \frac{1}{v(G_n)} \ln Z_{RC}(q,w) \to 0.$$

Theorem (BBCs '25+)

$$\mathbb{P}_{pIS}(A_n) < e^{-cv(G_n)} \implies \mathbb{P}_{cRC}(A_n) < e^{-c'v(G_n)}.$$

Lemma

For all $\varepsilon > 0$ and r there exists $c > 0$ such that

$$\mathbb{P}_{\sigma_n \sim pIS}\left[d_{TV}\left(S_r\sigma_n, \mu|_{B(o,r)}\right) \geq \varepsilon\right] < e^{-cv(G_n)}.$$

Thanks for your attention!