Graph limit theory and partition functions

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- **dense graph limits:** homomorphism densities, which can be seen as **partition functions**
- **sparse graph limits:** local neighborhood statistics
- **action convergence:** intermediate density, with applications to random matrices
- **entropy inequalities:** based on counting – connections to **partition functions**

Overview of literature

László Lovász: Large networks and graph limits, 2012, AMS.

Borgs, C., Chayes, J. T., Lovász, L., Sós, V. T., & Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. Advances in Mathematics, 219(6), 1801-1851.

Barvinok, A., & Soberón, P. (2017). Computing the **partition function for graph homomorphisms**. Combinatorica, 37, 633-650.

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Ágnes Backhausz, Balázs Szegedy, Action convergence of operators and graphs. Canadian Journal of Mathematics. 74 (1), 72-121 (2022).

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Question: given a growing sequence graphs, is there a **continuous limit object** representing structural properties of this sequence?

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- when do we say that a sequence of finite graphs converges?
- for a convergent sequence, is there a limit object?
- if we find the limit object and understand it with analytic tools, how can we translate the results back to the finite graphs?

Limits of dense graphs

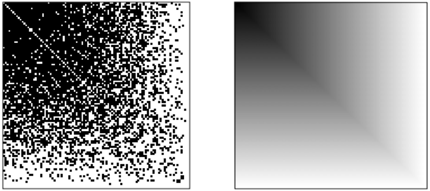

FIGURE 1.8. A randomly grown uniform attachment graph with 100 nodes, and a (continuous) function approximating it

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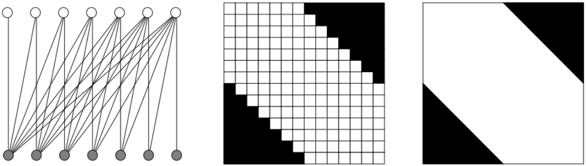

FIGURE 1.7. A half-graph, its pixel picture, and the limit function

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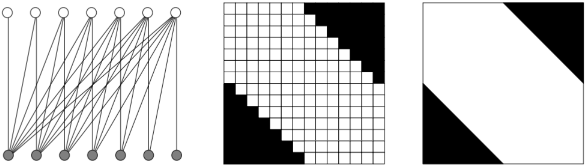

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Similarities:

- edge density: the probability that two randomly chosen vertices are connected
- triangle density: the probability that three randomly chosen vertices form a triangle

Similarity based on counting

Definition

Let H, G be two graphs. A map $f : V(H) \to V(G)$ is called a **graph homomorphism** if

- $f \times f : V(H) \times V(H) \to V(G) \times V(G)$ takes edges to edges;
- that is, for $(u, v) \in E(H)$ we have $(f(u), f(v)) \in E(G)$.

Let $\hom(H, G)$ denote the set of homomorphisms from H to G and let

$$t(H, G) := \frac{|\hom(H, G)|}{|V(G)|^{|V(H)|}}$$

We have that $t(H, G)$ is the probability that random map from $V(H)$ to $V(G)$ is a graph homomorphism. In particular $0 \leq t(H, G) \leq 1$.

Examples: if H is an edge, $t(H, G)$ is the edge density. If H is a triangle, $t(H, G)$ is the probability that three randomly chosen vertices form a triangle.

Limit objects in dense graph limit theory

Definition

A **graphon** is a measurable function $W : [0,1]^2 \to [0,1]$ such that $W(x,y) = W(y,x)$ holds for every $x, y \in [0,1]$.

If H is a finite graph with $V(H) = \{1, 2, \dots, n\}$, then we define

$$t(H, W) := \int_{x_1, x_2, \dots, x_n \in [0,1]} \prod_{(i,j) \in E(H)} W(x_i, x_j) \, dx_1 dx_2 \dots dx_n.$$

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Theorem (Lovász–Szegedy, 2006)

If G_n is **convergent** *in the sense that $t(H, G_n)$ is convergent for every simple finite graph H, then there is a graphon W such that $\lim_{n\to\infty} t(H, G_n) = t(H, W)$ holds for every finite graph H.*

Example: growing Erdős–Rényi graphs with edge probability p converge to the constant p graphon (each pair of vertices is connected independently with probability p).

Graph homomorphism partition function

Definition

Let $H = (V, E)$ be a finite simple graph, and $A \in \mathbb{C}^{k \times k}$ be a symmetric matrix. The **graph homomorphism partition function** is defined by

$$P_H(A) = \sum_{\phi : V \to \{1,2,\ldots,k\}} \prod_{\{u,v\} \in E} A_{\Phi(u)\Phi(v)}.$$

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If A is the adjacency matrix of a graph G, then this is the same as the number of homomorphisms from H to G.

With other choices of A, we can get the number of colorings, number of independent sets etc.

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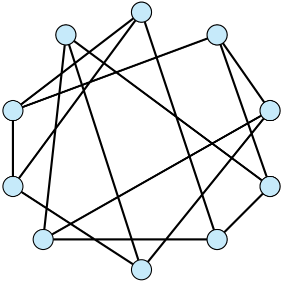
- a random 3-regular graph on 1000 vertices and a random 3-regular graph on 10000 vertices: they are similar **locally**: the small neighborhood of a randomly chosen vertex is a tree with high probability

Random regular graphs

$G = G(n, d)$: a uniformly chosen, simple d-regular graph on n vertices.



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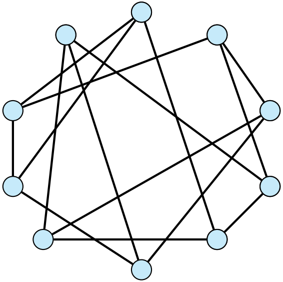

Local properties: the graph locally looks like a tree $\Rightarrow$ the **local limit** will be the infinite d-regular tree, if d **is fixed** and $n \to \infty$.

Random regular graphs

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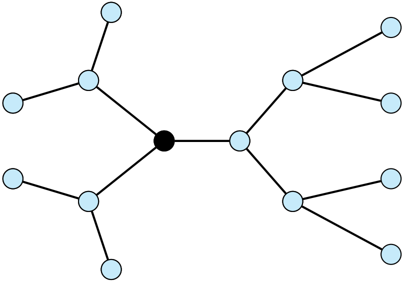


Random regular graphs

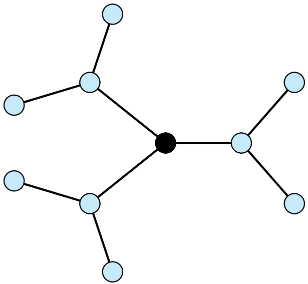
Fix $d \geq 3$.

$G = G(n, d)$: a uniformly chosen, simple d-regular graph on n vertices.

Local properties: $G(n, d)$ does not contain many small cycles with high probability – it looks like a tree.

$G(n, d)$ tends to the **infinite d-regular tree T_d** in the Benjamini–Schramm (local) sense:

given n and r, the probability that the r-neighborhood of a uniformly chosen random vertex is a tree, tends to 1 as $n \to \infty$.



Limits of colored regular graphs

S: finite set of colors

(H_n): a sequence of finite **d-regular graphs with colored vertices** (with the number of vertices tending to infinity but all degrees bounded by Δ)

$\mathcal{F}(\Delta, r, S)$: the set of connected rooted vertex-colored graphs with diameter at most $2r$

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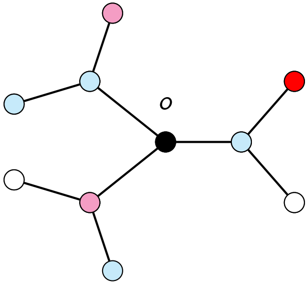
We say that (H_n) **converges locally** to $(X_v)_{v \in T_d}$ if for every r and $F \in \mathcal{F}(\Delta, r, S)$ the following holds. The probability that the colored rooted r-neighborhood of a uniformly chosen vertex v of H_n is isomorphic to F converges to the probability that the colored r-neighborhood of the root o of T_d is isomorphic to F.

Limits of colored regular graphs

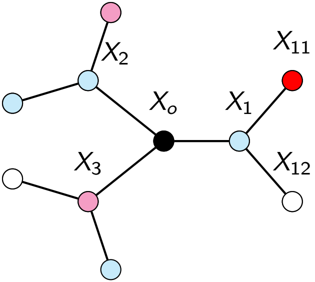
$r = 2$, F as below with the black vertex as the root:



The probability that the 2-neighborhood of a randomly chosen vertex is isomorphic to F should be convergent.

Limits of colored regular graphs

2-neighborhood of the root in an invariant random process:



$X_o, X_1, X_2, \ldots$ are random colors from S.

Typical processes

T_d: infinite d-regular tree, S: finite set

Definition (Typical process)

We say that an S-valued invariant random process $(X_v)_{v \in V(T_d)}$ is **typical** *if there exists a subsequence of the positive integers (n_k) with the following property.*

If, for each k independently, G_k is a random d-regular graph on n_k vertices, then, with probability 1, there exists a sequence of colorings $f_k : V(G_k) \to S$ such that (G_k, f_k) converges to $(X_v)_{v \in V(T_d)}$ locally as $k \to \infty$.

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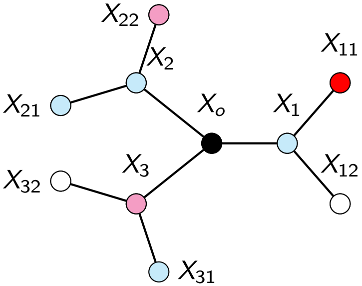
Open question: do we need subsequence in this definition?

Example for not typical process: alternating black and white with the color of the root chosen uniformly at random (Bollobás, 1984: a random d-regular graph is far from being bipartite with high probability, its independence ratio is smaller than $1/2$).

Entropy inequalities

Let $U \subset V(T_d)$ be a finite connected subgraph of the infinite tree. Then the entropy of the joint distribution $\underline{X} = (X_v)_{v \in U}$ will be denoted by $h(U)$:

$$h(U) = -\sum_F \mathbb{P}(\underline{X} = F) \cdot \log \mathbb{P}(\underline{X} = F).$$



Example: $h(B_2(o)) = h(X_o, X_1, X_2, X_{11}, X_{12}, \ldots, X_{32})$, where $B_2(o)$ is the 2-neighborhood of the root.

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Proposition

For every typical process the following hold:

(i)

$$\frac{d}{2} h(\bullet\!-\!\bullet) \geq (d-1) h(\bullet).$$

(ii)

$$h(B_1(\bullet)) \geq \frac{d}{2} h(\bullet\!-\!\bullet),$$

where $B_1(\bullet)$ is the 1-neighborhood of a vertex (a vertex and its d neighbors).

For factor of i.i.d. processes: Bowen (2008); the f-invariant is nonnegative; see also Rahman–Virág (2014).

Entropy inequalities

Proposition

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Idea of the proof (similar to Bollobás's argument for the independence ratio):

- take the configuration model of the random regular graph;
- **count the number of colorings that are close to the distribution of X_v on $B_1(\cdot)$**;
- this is more than the total number of graphs.

Testing a matrix with a vector

Let $M \in \mathbb{R}^{n \times n}$ be a matrix. We consider every row vector $v \in \mathbb{R}^n$ as an "experiment" that we can perform on M.

1. Let $w := vM$.
2. **Joint empirical distribution:** let $\mu_{v,M}$ denote the distribution of $(v(i), w(i))$, where i is picked uniformly at random from $\{1, 2, \ldots, n\}$.

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Very rough idea: Two matrices $N \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{m \times m}$ are considered to be similar if the set of all possible observations on them are similar.

Extensions to random matrices (work in progress)

Typical probability measure: can be approximated with the empirical distribution of $(v, vA, vA^2, \ldots, vA^{k-1})$ with an appropriate v with high probability if A is chosen randomly

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Theorem (B-Szegedy, 2025+)

Let ν be a typical probability measure on $\mathbb{C}^k$ with finite covariance matrix and $\|\nu^{[1]}\|_p < \infty$ for some $p > 1$. Let $\sigma > 0$ and N_k be the standard normal distribution on $\mathbb{C}^k$ (for $k \geq 1$). Then we have

$$\mathbb{D}(\nu \star \sigma N_k) + \int_{\mathbb{C}^{k-1}} \log \varphi^{(\sigma)}_{1,\dots,k-1} d\nu^{(\sigma)}_{2,\dots,k} \geq \mathbb{D}(\sigma N_1), \tag{1}$$

where $\nu^{(\sigma)} = \nu \star \sigma N_k$ and $\varphi^{(\sigma)}$ denotes the density of the Gaussian distribution having the same mean and covariance structure as $\nu^{(\sigma)}$.